MATHEMATICS I & II

DIPLOMA COURSE IN ENGINEERING

FIRST SEMESTER

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FOREWORD

We take great pleasure in presenting this book of mathematics to the students of Polytechnic Colleges. This book is prepared in accordance with the new syllabus framed by the Directorate of Technical Education, Chennai.

This book has been prepared keeping in mind, the aptitude and attitude of the students and modern methods of education. The lucid manner in which the concepts are explained, make the teaching learning process more easy and effective. Each chapter in this book is prepared with strenuous efforts to present the principles of the subject in the most easy-to-understand and the most easy-to-workout manner.

Each chapter is presented with an introduction, definition, theorems, explanation, worked examples and exercises given are for better understanding of concepts and in the exercises, problems have been given in view of enough practice for mastering the concept.

We hope that this book serves the purpose i.e., the curriculum which is revised by DTE, keeping in mind the changing needs of the society, to make it lively and vibrating. The language used is very clear and simple which is up to the level of comprehension of students.

List of reference books provided will be of much helpful for further reference and enrichment of the various topics.

We extend our deep sense of gratitude to Thiru.S.Govindarajan, Co-ordinator and Principal, Dr. Dharmambal Government polytechnic College for women, Chennai and Thiru. P.L. Sankar, convener, Rajagopal polytechnic College, Gudiyatham who took sincere efforts in preparing and reviewing this book.

Valuable suggestions and constructive criticisms for improvement of this book will be thankfully acknowledged.

Wishing you all success.

Authors

SYLLABUS

FIRST SEMESTER MATHEMATICS - I

UNIT - I

DETERMINANTS

1.1 Definition and expansion of determinants of order 2 and 3 .Properties of determinants .Cramer's rule to solve simultaneous equations in 2 and 3 unknowns-simple problems.

1.2 Problems involving properties of determinants

1.3 Matrices :Definition of matrix .Types of matrices .Algebra of matrices such as equality, addition, subtraction, scalar multiplication and multiplication of matrices. Transpose of a matrix, adjoint matrix and inverse matrix-simple problems.

UNIT - II

BINOMIAL THEOREM

2.1 Definition of factorial notation, definition of Permutation and Combinations with formula. Binomial theorem for positive integral index (statement only), finding of general and middle terms. Simple problems.

2.2 Problems finding co-efficient of x^n, independent terms. Simple problems. Binomial Theorem for rational index, expansions, only upto – 3 for negative integers. Simple Expansions

2.3 Partial Fractions :Definition of Polynomial fraction, proper and improper fractions and definition of partial fractions.

To resolve proper fraction into partial fraction with denominator containing non repeated linear factors, repeated linear factors and irreducible non repeated quadratic factors. Simple problems.

UNIT - III

STRAIGHT LINES

3.1 Length of perpendicular distance from a point to the line and perpendicular distance between parallel lines. Simple problems.

Angle between two straight lines and condition for parallel and perpendicular lines. Simple problems

3.2 Pair of straight lines Through origin :Pair of lines passing through the origin $ax^2+2hxy+by^2=0$ expressed in the form $(y-m_1x)(y-m_2x) =0$. Derivation of angle between pair of straight lines. Condition for parallel and perpendicular lines. Simple problems

3.3 Pair of straight lines not through origin: Condition for general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent pair of lines.(Statement only) Angle between them, condition for parallel and perpendicular lines. Simple problems.

UNIT - IV

TRIGONOMETRY

4.1 Trigonometrical ratio of allied angles-Expansion of Sin(A+B) and cos(A+B)- problems using above expansion

4.2 Expansion of tan(A+B) and Problems using this expansion

4.3 Trigonometrical ratios of multiple angles (2A only) and sub-multiple angles. Simple problems.

UNIT - V

TRIGONOMETRY

5.1 Trigonometrical ratios of multiple angels (3A only) Simple problems.

5.2 Sum and Product formulae-Simple Problems.

5.3 Definition of inverse trigonometric ratios, relation between inverse trigonometric ratios-Simple Problems

FIRST SEMESTER MATHEMATICS II

UNIT - I

CIRCLES

1.1 Equation of circle – given centre and radius. General Equation of circle – finding center and radius. Simple problems.

1.2 Equation of circle through three non collinear points – concyclic points. Equation of circle on the line joining the points (x_1,y_1) and (x_2,y_2) as diameter. Simple problems.

1.3 Length of the tangent-Position of a point with respect to a circle. Equation of tangent (Derivation not required). Simple problems.

UNIT-II

FAMILY OF CIRCLES:

2.1 Concentric circles – contact of circles (internal and external circles) – orthogonal circles – condition for orthogonal circles.(Result only). Simple Problems

2.2 **Limits:** Definition of limits -

$$\underset{x \to a}{\mathrm{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\underset{\theta \to 0}{\mathrm{Lt}} \frac{\sin\theta}{\theta} = 1, \qquad \underset{\theta \to 0}{\mathrm{Lt}} \frac{\tan\theta}{\theta} = 1 \qquad (\theta \text{ in radian})$$

[Results only] – Problems using the above results.

2.3 **Differentiation:** Definition – Differentiation of u/v x^n, sinx, cosx, tanx, cotx, secx, cosecx, logx, e^x, u v, uv, uvw, (Results only). Simple problems using the above results.

UNIT- III

3.1 Differentiation of function of functions and Implicit functions. Simple Problems.

3.2 Differentiation of inverse trigonometric functions and parametric functions. Simple problems.

3.3 Successive differentiation up to second order (parametric form not included) Definition of differential equation, formation of differential equation. Simple Problems

UNIT- IV

APPLICATION OF DIFFERENTIATION–I

4.1 Derivative as a rate measure-simple Problems.

4.2 Velocity and Acceleration-simple Problems

4.3 Tangents and Normals-simple Problems

UNIT-V

APPLICATION OF DIFFERENTIATION –II

5.1 Definition of Increasing function, Decreasing function and turning points. Maxima and Minima (for single variable only) – Simple Problems.

5.2 **Partial Differentiation:** Partial differentiation of two variable up to second order only. Simple problems.

5.3 Definition of Homogeneous functions-Eulers theorem-Simple Problems.

FIRST SEMESTER

MATHEMATICS - I

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SEMESTER I

MATHEMATICS – I

UNIT – I

DETERMINANTS

1.1 Definition and expansion of determinants of order 2 and 3 Properties of determinants Cramer's rule to solve simultaneous equations in 2 and 3 unknowns-simple problems.

1.2 Problems involving properties of determinants

1.3 **Matrices**

Definition of matrix. Types of matrices. Algebra of matrices such as equality, addition, subtraction, scalar multiplication and multiplication of matrices. Transpose of a matrix, adjoint matrix and inverse matrix-simple problems.

1.1. DETERMINANT

The credit for the discovery of the subject of determinant goes to the German mathematician, Gauss. After the introduction of determinants, solving a system of simultaneous linear equations becomes much simpler.

Definition:

Determinant is a square arrangement of numbers (real or complex) within two vertical lines.

Example :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ is a determinant}$$

Determinant of second order:

The symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ consisting of 4 numbers a, b, c and arranged in two rows and two columus is called a determinant of second order.

The numbers a,b,c, and d are called elements of the determinant The value of the determinant is Δ = ad - bc

Examples:

1. $\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$ = (2) (1) – (5) (3) = 2 – 15 = -13

2. $\begin{vmatrix} 4 & 6 \\ 3 & -5 \end{vmatrix}$ = (4) (-5) – (6) (3) = -20 – 18 = -38

Determent of third order:

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of nine elements arranged in three rows and three columns is called a determinant of third order

The value of the determinant is obtained by expanding the determinant along the first row

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

Note: The determinant can be expanded along any row or column.

Examples:

(1) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 5 & 2 & 1 \end{vmatrix}$

$$\begin{aligned} &= 1(1-8) - 2(2-20) + 3(4-5) \\ &= 1(-7) - 2(-18) + 3(-1) \\ &= -7 + 36 - 3 \\ &= -10 + 36 = 26 \end{aligned}$$

(2) $\begin{vmatrix} 3 & 0 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & -2 \end{vmatrix}$

$$\begin{aligned} &= 3(6+4) + 1(-2+3) \\ &= 3(10) + 1(1) \\ &= 30 + 1 \\ &= 31 \end{aligned}$$

Minor of an element

Definition :

Minor of an element is a determinant obtained by deleting the row and column in which that element occurs. The Minor of I^{th} row J^{th} Column element is denoted by m_{ij}

Example:

$$\begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \\ 11 & 5 & -3 \end{vmatrix}$$

Minor of 3 = $\begin{vmatrix} 0 & 4 \\ 11 & 5 \end{vmatrix}$ = 0-44 = -44

Minor of 0 = $\begin{vmatrix} -1 & 3 \\ 5 & -3 \end{vmatrix}$ = 3-15 = -12

Cofactor of an element

Definition :

Co-factor of an element in i^{th} row,j^{th} column is the signed minor of I^{th} row J^{th} Column element and is denoted by A_{ij}.

(i.e) $A_{ij} = (-1)^{i+j}\, m_{ij}$

The sign is attached by the rule $(-1)^{i+j}$

Example

$$\begin{vmatrix} 3 & -2 & 4 \\ 2 & 1 & 0 \\ 7 & 11 & 6 \end{vmatrix}$$

Co-factor of -2 = $(-1)^{1+2} \begin{vmatrix} 2 & 0 \\ 7 & 6 \end{vmatrix}$ = $(-1)^3$ (12)= -12

Co-factor of 7= $(-1)^{3+1} \begin{vmatrix} -2 & 4 \\ 1 & 0 \end{vmatrix}$ = $(-1)^4$ (0-4)= -4

Properties of Determinants:

Property 1:

The value of a determinant is unaltered when the rows and columns are interchanged.

(i.e) If $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta^T = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

then $\Delta^T = \Delta$

Property 2:

If any two rows or columns of a determinant are interchanged the value of the determinant is changed in its sign.

If $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

then $\Delta_1 = -\Delta$

Note: R_1 and R_2 are interchanged.

Property 3:

If any two rows or columns of a determinant are identical, then the value of the determinant is zero.

(i.e) The value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is zero Since $R_1 \equiv R_2$

Property 4:

If each element of a row or column of a determinant is multiplied by any number $K \neq 0$, then the value of the determinant is multiplied by the same number K.

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and $\Delta_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

then $\Delta_1 = K\Delta$

Property 5:

If each element of a row or column is expressed as the sum of two elements, then the determinant can be expressed as the sum of two determinants of the same order.

(i.e) If $\Delta = \begin{vmatrix} a_1+d_1 & b_1+d_2 & c_1+d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

then $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{bmatrix} d_1 & d_2 & d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Property 6:

If each element of a row or column of a determinant is multiplied by a constant $K \neq 0$ and then added to or subtracted from the corresponding elements of any other row or column then the value of the determinant is unaltered.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta_1 = \begin{vmatrix} a_1+ma_2+na_3 & b_1+mb_2+nb_3 & c_1+mc_2+nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ma_2 & mb_2 & mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} na_3 & nb_3 & nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \Delta + m \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + n \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\Delta_1 = \Delta + m\,(0) + n\,(0) = \Delta$

Property 7:

In a given determinant if two rows or columns are identical for x = a, then (x-a) is a factor of the determinant.

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$\text{For } a=b, \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ b & b & c \\ b^3 & b^3 & c^3 \end{vmatrix} = 0 \qquad [C_1, \text{ and } C_2 \text{ are identical}]$$

$\therefore$ (a-b) is a factor of Δ

Notation :

Usually the three rows of the determinant first row, second row and third row are denoted by R_1, R_2 and R_3 respectively and the columns by C_1, C_2 and C_3

If we have to interchange two rows say R_1 and R_2 the symbol double sided arrow will be used. We will write like this $R_2 \leftrightarrow R_2$ it should be read as "is interchanged with" similarly for columns $C_2 \leftrightarrow C_2$.

If the elements of R_2 are subtracted from the corresponding elements of R_1, then we write $R_1 - R_2$ similarly for columns also.

If the elements of one column say C_1,'m' times the element of C_2 and n times that of C_3 are added, we write like this $C_1 \rightarrow C_1 + m C_2 + n C_3$. Here one sided arrow is to be read as "is changed to"

Solution of simultaneous equations using Cramer's rule:

Consider the linear equations**.**

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

$$\text{let } \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Then $x = \dfrac{\Delta_x}{\Delta}$ and $y = \dfrac{\Delta_y}{\Delta}$, provided $\Delta \neq 0$

x, y are unique solutions of the given equations. This method of solving the line equations is called Cramer's rule.

Similarly for a set of three simultaneous equations in x, y and z

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$ and

$a_3 x + b_3 y + c_3 z = d_3$, the solution of the system of equations, by cramer's rule is given by, $x = \dfrac{\Delta_x}{\Delta}$, $y = \dfrac{\Delta_y}{\Delta}$ and $z = \dfrac{\Delta_z}{\Delta}$,

provided $\Delta \neq 0$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

1.1 WORKED EXAMPLES

PART – A

1. Solve $\begin{vmatrix} x & 2 \\ x & 3x \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} x & 2 \\ x & 3x \end{vmatrix} = 0$$

$$3x^2 - 2x = 0$$

$$x(3x - 2) = 0$$

$$x = 0 \text{ or } x = \frac{2}{3}$$

2. Solve $\begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = 0$$

$$x^2 - 16 = 0$$

$$x^2 = 16$$

$$x = \pm 4$$

3. Find the value of 'm' when $\begin{vmatrix} m & 2 & 1 \\ 3 & 4 & 2 \\ -7 & 3 & 0 \end{vmatrix} = 0$

Solution:

Given $\begin{vmatrix} m & 2 & 1 \\ 3 & 4 & 2 \\ -7 & 3 & 0 \end{vmatrix} = 0$

Expanding the determinant along, R_1 we have

m(0-6)-2 (0+14) +1 (9+28) = 0

m(-6) -2 (14) +1 (37) = 0

$-6m - 28 + 37 = 0$

$-6m + 9 = 0$

$-6m = -9$

$$m = \frac{9}{6} = \frac{3}{2}$$

4. Find the Co-factor of element 3 in the determinant $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix}$

Solution:

Cofactor of 3 = $A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 5 & 7 \end{vmatrix}$

$= (-1)^4 (7-0) = 7$

PART – B

1. Using cramer's rule, solve the following simultaneous equations

$x + y + z = 2$

$2x - y - 2z = -1$

$x - 2y - z = 1$

Solution:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$$

$= 1 (1-4) -1 (-2+2) +1 (-4+1)$

$= 1 (-3) -1 (0) +1 (-3)$

$= -3 -3 = -6 \neq 0$

$$\Delta_x = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$$

$= 2\ (1-4)\ -1\ (1+2)\ +1\ (2+1)$

$= 2\ (-3)\ -1\ (3)\ +1\ (3)$

$= -6\ -3\ +3 = -6$

$$\Delta_y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 1\ (1+2)\ -2\ (-2+2)\ +1\ (2+1)$

$= 1\ (3)\ -2\ (0)\ +1\ (3)$

$= 3+3 = 6$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{vmatrix}$$

$= 1(-1-2)\ -1\ (2+1)\ +2\ (-4+1)$

$= -3-3-6 = -12$

$\therefore$ By Cramer's rule,

$$x = \frac{\Delta_x}{\Delta} = \frac{-6}{-6} = 1 \qquad y = \frac{\Delta_y}{\Delta} = \frac{6}{-6} = -1$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-12}{-6} = 2$$

2. Using Cramer's rule solve: -2y+3z-2x+1=0

-x+y-z+5=0 -2z -4x+y = 4

Solution:

Rearrange the given equations in order

-2x-2y+3z = -1; -x+y-z = -5; -4x+y-2z =4

$$\Delta = \begin{vmatrix} -2 & -2 & 3 \\ -1 & 1 & -1 \\ -4 & 1 & -2 \end{vmatrix}$$

$= -2(-2+1) +2 (2-4) + 3(-1+4)$

$= -2 (-1) +2 (-2) +3 (3)$

$= 2-4+9 =7$

$$\Delta_x = \begin{vmatrix} -1 & -2 & 3 \\ -5 & 1 & -1 \\ 4 & 1 & -2 \end{vmatrix}$$

$= -1(-2+1) +2 (10+4) +3 (-5-4)$

$= 1+28-27$

$= 2$

$$\Delta_y = \begin{vmatrix} -2 & -1 & 3 \\ -1 & -5 & -1 \\ -4 & 4 & -2 \end{vmatrix}$$

$= -2 (10+4) +1 (2-4) +3 (-4-20)$

$= -2(14) +1 (-2) +3 (-24)$

$= -28-2-72$

$= -102$

$$\Delta_z = \begin{vmatrix} -2 & -2 & -1 \\ -1 & 1 & -5 \\ -4 & 1 & 4 \end{vmatrix}$$

$= -2 (4+5) +2 (-4-20) -1 (-1+4)$

$= -18-48-3$

$= -69$

$$x = \frac{\Delta_x}{\Delta} = \frac{2}{7}, y = \frac{\Delta_y}{\Delta} = \frac{-102}{7}, \text{ and } z = \frac{\Delta_z}{\Delta} = \frac{-69}{7}$$

3. Using Cramer's rule solve

$2x-3y =5$

$x-8=4y$

Solution:

2x-3y = 5

x-4y = 8

$\Delta = \begin{vmatrix} 2 & -3 \\ 1 & -4 \end{vmatrix}$ = (2) (-4) – (-3) (1)

= -8 +3= -5

$\Delta_x = \begin{vmatrix} 5 & -3 \\ 8 & -4 \end{vmatrix}$ = (5) (-4) – (-3) (8)

= -20 + 24= 4

$\Delta_y = \begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix}$ = 16 – 5= 11

By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = \frac{4}{-5} = -\frac{4}{5}$$

$$y = \frac{\Delta_y}{\Delta} = \frac{11}{-5} = -\frac{11}{5}$$

1.2 PROBLEMS INVOLVING PROPERTIES OF DETERMINANTS

PART-A

1) Evaluate

$$\begin{vmatrix} 20 & 11 & 31 \\ 11 & -7 & 4 \\ 19 & 11 & 30 \end{vmatrix}$$

Solution:

$$\Delta = \begin{vmatrix} 20 & 11 & 31 \\ 11 & -7 & 4 \\ 19 & 11 & 30 \end{vmatrix}$$

$$= \begin{vmatrix} 31 & 11 & 31 \\ 4 & -7 & 4 \\ 30 & 11 & 30 \end{vmatrix} \quad C_1 \to C_1 + C_2$$

$\Delta = 0$ since $C_1 \equiv C_3$

2) Without expanding, find the value of

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & -6 & 9 \end{vmatrix}$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & -6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 3(1) & 3(-2) & 3(3) \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 1 & -2 & 3 \end{vmatrix}$$

$= 3\ (0) = 0$, since $R_1 \equiv R_3$

3) Evaluate

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix} \qquad C_2 \to C_2 + C_3$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$= (a+b+c)\ (0) = 0$, since $C_1 \equiv C_2$

4) Prove that $\begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} = 0$

Solution:

$$\text{L.H.S} = \begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix}$$

$$\begin{vmatrix} x-y+y-z+z-x & y-z & z-x \\ y-z+z-x+x-y & z-x & x-y \\ z-x+x-y+y-z & x-y & y-z \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 0 & y-z & z-x \\ 0 & z-x & x-y \\ 0 & x-y & y-z \end{vmatrix} = 0 = \text{R.H.S}$$

PART – B

1) Prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ = (x-y) (y-z) (z-x)

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix} \qquad R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$$

$$= (x-y)\,(y-z) \begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$

$$= (x-y)\,(y-z) \begin{vmatrix} 1 & x+y \\ 1 & y+z \end{vmatrix} \text{ (expanded along the first column)}$$

$$=(x-y)\,(y-z)\,[1(y+z) - 1(x+y)]$$

$$= (x-y)\,(y-z)\,(z-x)$$

L.H.S = R.H.S

2) Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a + b + c)\,(a-b)\,(b-c)\,(c-a)$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} o & o & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} \qquad C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$\Delta = (a-b)\,(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$\Delta = (a-b)\,(b-c) \begin{vmatrix} 1 & 1 \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$

(expanded along the first row)

$$= (a-b)(b-c)\left[b^2+bc+c^2-\left(a^2+ab+b^2\right)\right]$$

$$= (a-b)(b-c)[b^2+bc+c^2-a^2-ab-b^2]$$

$$= (a-b)(b-c)[bc+c^2-a^2-ab]$$

$$= (a-b)(b-c)[c^2-a^2+b(c-a)]$$

$$= (a-b)(b-c)[(c+a)(c-a)+b(c-a)]$$

$$= (a-b)(b-c)(c-a)[c+a+b] \qquad = R.H.S$$

3) Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2(3+a)$

Solution:

$$L.H.S = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$\begin{vmatrix} 3+a & 3+a & 3+a \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} \qquad R_1 \to R_1 + R_2 + R_3$$

$$= 3+a\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (3+a)\begin{vmatrix} 0 & 0 & 1 \\ -a & a & 1 \\ 0 & -a & 1+a \end{vmatrix} \quad C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$$

$$= (3+a)\begin{vmatrix} -a & a \\ 0 & -a \end{vmatrix} = (3+a)(a^2-0)$$

$$= a^2(3+a) \qquad = R.H.S$$

4) Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad R_1 \to R_1 + R_2 + R_3$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix} \quad \begin{matrix} C_1 \to C_1 - C_2, \\ C_2 \to C_2 - C_3 \end{matrix}$$

$$= (a+b+c) \begin{vmatrix} a+b+c & -(a+b=c) \\ 0 & (a+b+c) \end{vmatrix}$$

$$= (a+b+c)\,[(a+b+c)^2 - 0]$$

$$= (a+b+c)^3 \qquad = \text{R.H.S}$$

5) Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left[1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right]$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} a(\frac{1}{a}+1) & a(\frac{1}{a}) & a(\frac{1}{a}) \\ b(\frac{1}{b}) & b(\frac{1}{b}+1) & b(\frac{1}{b}) \\ c(\frac{1}{c}) & c(\frac{1}{c}) & c(\frac{1}{c}+1) \end{vmatrix}$$

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$=abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad R_1 \to R_1 + R_2 + R_3$$

$$= abc \ (1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc \ (1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & \frac{1}{c}+1 \end{vmatrix} \quad C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$$

$$= abc \ (1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= abc \ (1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \ [1-0]$$

$$\Delta = abc \ (1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})$$

1.3 MATRICES

Introduction:

The term matrix was first introduced by a French mathematician Cayley in the year 1857. The theory of matrices is one of the powerful tools of mathematics not only in the field of higher mathematics but also in other branches such as applied sciences, nuclear physics, probability and statistics, economics and electrical circuits.

Definition:

A Matrix is a rectangular array of numbers arranged in to rows and columns enclosed by parenthesis or square brackets.

Example:

1. A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2.** B = $\begin{pmatrix} 2 & 1 & 0 \\ -5 & 6 & 7 \\ 1 & 0 & 8 \end{pmatrix}$

Usually the matrices are denoted by capital letters of English alphabets A,B,C…,etc and the elements of the matrices are represented by small letters a,b,c,.etc.

Order of a matrix

If there are m rows and n columns in a matrix, then the order of the matrix is mxn or m by n.

Example: A = $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

A has two rows and three columns. We say that A is a matrix of order 2x3

Types of matrices

Row matrix:

A matrix having only one row and any number of columns is called a row matrix.

Eg: A = (1 2 -3)

Column matrix

Matrix having only one column and any number of rows is called a column matrix.

Eg: $B = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$

Square matrix

A matrix which has equal number of rows and columns is called a square matrix.

Eg: $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 4 \\ 3 & 2 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 3 & -9 \\ 4 & 1 \end{bmatrix}$

A is a square matrix of order 3

B is a square matrix of order 2

Null matrix (or) zero matrix or, void matrix:

If all the elements of a matrix are zero, then the matrix is called a null or zero matrix or void matrix it is denoted by O.

Eg: $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $(2) O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Diagonal matrix:

A square matrix with all the elements equal to zero except those in the leading diagonal is called a diagonal matrix

Eg: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Unit matrix:

Unit matrix is a square matrix in which the principal diagonal elements are all ones and all the other elements are zeros.

It is denoted by I.

$$\text{Eg: } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here I_3 is a unit matrix of order 3.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I_2 is a unit matrix of order 2.

Algebra of matrices:

Equality of two matrices:

Two matrices A and B are said to be equal if and only if order of A and order of B are equal and the corresponding elements of A and B are equal.

$$\text{Eg: if } A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$$

then A = B means $a=1 \;\; b=0 \;\; c=5$

$x=-3 \;\; y=1 \;\; z=4$

Addition of matrices:

If A and B are any two matrices of the same order, then their sum A+B is of the same order and is obtained by adding the corresponding elements of A and B.

$$\text{Eg: If } A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \text{ then}$$

$$A+B = \begin{bmatrix} 1+4 & 2+6 \\ 3+7 & 0+9 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 9 \end{bmatrix}$$

Note : If the matrices are different order, addition is not possible.

Subtraction of matrices:

If A and B are any two matrices of the same order, then their difference A-B is of the same order and is obtained by subtracting the elements of B from the corresponding elements of A.

Eg: If A= $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ B = $\begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$ then

$$A-B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1-4 & 3-1 \\ 2-2 & 0+1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & 1 \end{bmatrix}$$

Scalar multiplication of a matrix

If A is a given matrix, K is a number real or complex and K≠0 then KA is obtained by multiplying each element of A by K. It is called scalar multiplication of the matrix.

Eg: if A = $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 5 \\ 3 & 1 & 6 \end{bmatrix}$ and K=3

$$KA = 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 5 \\ 3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -3 \\ 6 & 0 & 15 \\ 9 & 3 & 18 \end{bmatrix}$$

Multiplication of two matrices:

Two matrices A and B are conformable for multiplication if and only if the number of columns in A is equal to the number of rows in B.

Note: If A is mxn matrix and B is nxp matrix then AB exists and is of order mxp.

Method of multiplication:

Let A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}_{2x3}$ and B = $\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}_{3x3}$

Here the number of columns of matrix A is equal to the number of rows of matrix B. Hence AB can be found and the order is 2x3.

Each element of the first row of AB is got by adding the product of the elements of first row of A with the corresponding elements of first, second and third columns of B. On similar lines, we can also get the second row of AB.

(ie) AB = $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$

$$= \begin{pmatrix} a_1x_1 + b_1x_2 + c_1x_3 & a_1y_1 + b_1y_2 + c_1y_3 & a_1z_1 + b_1z_2 + c_1z_3 \\ a_2x_1 + b_2x_2 + c_2x_3 & a_2y_1 + b_2y_2 + c_2y_3 & a_2z_1 + b_2z_2 + c_2z_3 \end{pmatrix}$$

Note: (1) If A is of order 3x3 and B is of order 3x2 then AB is of order 3x2 but BA does not exist.

(2) If AB and BA are of same order they need not be equal. In general AB $\neq$ BA.

Example:

If A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 2 \\ 3 & 4 \\ 4 & -3 \end{bmatrix}$ then

$$AB = \begin{bmatrix} -1+6+12 & 2+8-9 \\ -3+0+8 & 6+0-6 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 5 & 0 \end{bmatrix}$$

Transpose of a matrix

If the rows and columns of a matrix are interchanged then the resultant matrix is called the transpose of the given matrix. It is denoted by A^T (or) A'

Example: If A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ 7 & 0 & 1 \end{bmatrix}$

then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & -5 & 0 \\ 3 & 6 & 1 \end{bmatrix}$

Note: (i) If a matrix A is of order mxn, then the order of A^T is nxm.

(ii) $(A^T)^T = A$

Co-factor matrix:

In a matrix, if all the elements are replaced by the corresponding co-factors is called the co-factor matrix.

Example:

The co-factor matrix of the matrix. $\begin{bmatrix} 1 & -4 \\ 8 & 3 \end{bmatrix}$ is as follows

Minors	co-factors
$m_{11} = 3$	$c_{11} = (-1)^{1+1}\, m_{11} = (-1)^2(3) = 3$
$m_{12} = 8$	$c_{12} = (-1)^{1+2}\, m_{12} = (-1)^3(8) = -8$
$m_{21} = -4$	$c_{21} = (-1)^{2+1}\, m_{21} = (-1)^3(-4) = 4$
$m_{22} = 1$	$c_{22} = (-1)^{2+2} m_{22} = (-1)^4(1) = 1$

co-factor matrix is $\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ 4 & 1 \end{bmatrix}$

Adjoint matrix (or) adjugate matrix:

The transpose of the co-factor matrix is called the adjoint matrix. or adjugate matrix. It is denoted by adj A.

Example

Let A = $\begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$

Cofactor of 3 $= (-1)^{1+1}(4) = 4$

Cofactor of 2 $= (-1)^{1+2}(-3) = 3$

Cofactor of -3 $= (-1)^{2+1}(2) = -2$

Cofactor of 4 $= (-1)^{2+2}(3) = 3$

$$\text{Cofactor matrix} = \begin{bmatrix} 4 & 3 \\ -2 & 3 \end{bmatrix}$$

$$\text{Adj A} = \begin{bmatrix} 4 & -2 \\ 3 & 3 \end{bmatrix}$$

Singular and Non-singular matrices:

A square matrix A is said to be singular if $|A| = 0$. If the determinant value of the square matrix A is not zero it is a non-singular matrix.

Example:

$$\text{Let A} = \begin{bmatrix} 2 & 5 & 7 \\ 7 & 1 & 6 \\ 5 & -4 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 5 & 7 \\ 7 & 1 & 6 \\ 5 & -4 & -1 \end{vmatrix}$$

$= 2\,(-1+24)-5(-7-30)+7(-28-5)$

$= 46+185-231 \qquad = 0$

$\therefore |A| = 0$

The given matrix A is singular

Inverse of a matrix:

Let A be a non-singular square matrix if there exists a square matrix B, such that AB=BA=I where I is the unit matrix of the same order as that of A, then B is called the inverse of matrix A and it is denoted by A^{-1}. (to be read as A inverse). This can be determined by using the formula. $A^{-1} = \dfrac{1}{|A|}$ adj A

Note:

1. if $|A|$ =0, then there is no inverse for the matrix
2. $A^{-1}A = AA^{-1} = I$,
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$

Working rule to find A^{-1}:

1) Find the determinant of A
2) Find the co-factor of all elements in A and form the co-factor matrix of A.
3) Find the adjoint of A.
4) $A^{-1} = \dfrac{\text{Adj}\,A}{|A|}$ provided $|A| \neq 0$

Note: For a second order matrix, the adjoint can easily be got by interchanging the principal diagonal elements and changing the signs of the secondary diagonal elements.

Example

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A| = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$$

$$\text{Adjoint } A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \therefore A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

WORKED EXAMPLES:

PART – A

1. If A = $\begin{pmatrix} 2 & 1 & -1 \\ 5 & 2 & 3 \end{pmatrix}$ what is the order of the matrix and find A^T

Solution :

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 5 & 2 & 3 \end{pmatrix}$$

The order of the matrix is 2x3

$$A^T = \begin{pmatrix} 2 & 5 \\ 1 & 2 \\ -1 & 3 \end{pmatrix}$$

2. If A = $\begin{pmatrix} 2 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 6 & 7 \end{pmatrix}$ find 3A – 2B

Solution:

$$3A-2B = 3\begin{pmatrix} 2 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix} - 2\begin{pmatrix} 3 & -1 & 4 \\ 2 & 6 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 9 & 0 \\ 15 & 6 & -3 \end{pmatrix} - \begin{pmatrix} 6 & -2 & 8 \\ 4 & 12 & 14 \end{pmatrix}$$

$$= \begin{pmatrix} 6-6 & 9+2 & 0-8 \\ 15-4 & 6-12 & -3-14 \end{pmatrix} = \begin{pmatrix} 0 & 11 & -8 \\ 11 & -6 & -17 \end{pmatrix}$$

3. If f(x) = 4x+2 and A = $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ find f(A)

Solution:

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} \text{and } f(x) = 4x + 2$$

$f(A) = 4A+2I_2$,

$$= 4\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -4 \\ 0 & 12 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$f(A) = \begin{pmatrix} 10 & -4 \\ 0 & 14 \end{pmatrix}$$

4. If X+Y = $\begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix}$ and X-Y = $\begin{pmatrix} 2 & 4 \\ 8 & 3 \end{pmatrix}$ find X and Y

solution:

Given X+Y = $\begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix}$ …….(1)

and X-Y = $\begin{pmatrix} 2 & 4 \\ 8 & 3 \end{pmatrix}$ ……….(2)

Adding 2X = $\begin{pmatrix} 8 & 8 \\ 10 & 4 \end{pmatrix}$ $\therefore$ X= $\frac{1}{2}\begin{pmatrix} 8 & 8 \\ 10 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 5 & 2 \end{pmatrix}$

Substitute matrix X in (1)

$$\begin{pmatrix} 4 & 4 \\ 5 & 2 \end{pmatrix} + Y = \begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 4 \\ 5 & 2 \end{pmatrix}$$

$$Y = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix}$$

5. Find the value of ‘a’ so that the

matrix $\begin{pmatrix} 1 & -2 & 0 \\ 2 & a & 4 \\ 2 & 1 & 1 \end{pmatrix}$ is singular

solution:

Let A = $\begin{pmatrix} 1 & -2 & 0 \\ 2 & a & 4 \\ 2 & 1 & 1 \end{pmatrix}$

The matrix A is singular, then $|A| = 0$

$$\begin{vmatrix} 1 & -2 & 0 \\ 2 & a & 4 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Expanding through first row

1 (a-4) +2 (2-8) = 0

a-4+4-16 = 0

a-16 = 0

a = 16

6. If A = $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix}$ find AB

Solution:

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0+0 & 3+0 \\ 0+2 & 6+4 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 10 \end{pmatrix}$$

7. Find A^2, if A = $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

Solution:

$$A^2 = A.A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4+1 & 2+0 \\ 2+0 & 1+0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

8. Find the adjoint of $\begin{pmatrix} 3 & 2 \\ -3 & 4 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} 3 & 2 \\ -3 & 4 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A_{11} = (-1)^{1+1}(4) = (-1)^2 4 = 4$$
$$A_{12} = (-1)^{1+2}(-3) = (-1)^3(-3) = 3$$
$$A_{21} = (-1)^{2+1}(2) = (-1)^3(2) = -2$$
$$A_{22} = (-1)^{2+2}(3) = (-1)^4(3) = 3$$

Cofactor matrix A = $\begin{pmatrix} 4 & 3 \\ -2 & 3 \end{pmatrix}$ $\therefore$ Adj A = $\begin{pmatrix} 4 & -2 \\ 3 & 3 \end{pmatrix}$

Aliter: Inter Changing elements in the principal diagonal and changing sign of elements in the other diagonal

Adj A = $\begin{pmatrix} 4 & -2 \\ 3 & 3 \end{pmatrix}$

9. Find the inverse of $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

Solution:

Step 1: let A = $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

Now $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

= 10-12

= -2≠0

$\therefore$ A^{-1} exists.

$$\text{adj } A = \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj} A = \frac{1}{-2} \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

PART-B

1) If $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & -4 \end{bmatrix}$

Show that AB = BA

Solution:

$$\text{Now } AB = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -4-1+0 & 2-2+0 & -2+2+0 \\ 0-2+2 & 0-4-1 & 0+4-4 \\ -2+0+2 & 1+0-1 & -1+0-4 \end{bmatrix}$$

$$AB = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4+0-1 & 2-2+0 & 0+1-1 \\ 2+0-2 & -1-4+0 & 0+2-2 \\ 4+0-4 & -2+2+0 & 0-1-4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$\therefore AB = BA$

2) Show that $AB \neq BA$ if $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$

Solution:

$$\text{Now } AB = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+2 & 2-2 \\ -3+4 & -2-4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 1 & -6 \end{pmatrix}$$

Similarly

$$BA = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2 & 6+8 \\ 1+1 & 2-4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 14 \\ 2 & -2 \end{pmatrix}$$

$\therefore AB \neq BA$

3) If $A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ Find $A^2 + 2A^T + I$

Solution:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

Now $A^2 + 2A^T + I$

$$= \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + 2\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6 & 2-8 \\ 3-12 & 6+16 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 4 & -8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 4 & -8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 \\ -5 & 15 \end{pmatrix}$$

4) If $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ show that (A- I) (A-4 I) =0

Solution:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A-I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A - 4I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$(A - I)(A - 4I) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad = O$$

5) If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ show that A^2-4A = 5I and hence find A^3

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

$$A^2 - 4A = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4 \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5I = \text{RHS}$$

To find A^3

We have proved that A^2-4A = 5I

$$A^2 = 4A+5I$$

Multiplying both sides by A, we get

$$A^3 = 4A^2+5AI = 4A^2+5A$$

$$= 4\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} +5\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{pmatrix} + \begin{pmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{pmatrix}$$

6) If $A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ Show that $(AB)^{-1} = B^{-1}A^{-1}$

Solution:

$$AB = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2+0 & 4+3 \\ 1+0 & -2+0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 7 \\ 1 & -2 \end{pmatrix}$$

$$|AB| = \begin{vmatrix} -2 & 7 \\ 1 & -2 \end{vmatrix} = 4-7 = -3 \neq 0$$

$(AB)^{-1}$ exists

$$\text{Adj }(AB) = \begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix}$$

$$(AB)^{-1} = \frac{Adj(AB)}{|AB|} = \frac{1}{-3}\begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix} \quad (1)$$

$$B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$|B| = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1-0 = -1 \neq 0$$

B^{-1} exist

$$Adj\ (B) = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$

$$B^{-1} = \frac{AdjB}{|B|} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$$

$|A| = \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 0+3 = 3 \neq 0$ $\quad A^{-1}$ exist

$$Adj\ A = \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{AdjA}{|A|} = \frac{1}{3}\begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$$

$$B^{-1}A^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix} \quad (2)$$

From (1) and (2)

$(AB)^{-1} = B^{-1}A^{-1}$

7. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$

Solution :

Let A = $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{vmatrix}$

= 1(3-0) -1 (6-0) -1 (4+1)

= 1(3) -1 (6) -1 (5)

= 3-6-5 = -8 $\neq$ 0 A^{-1} exists

$A_{11} = + \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$ = 3-0 =3

$A_{12} = - \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix}$ = - (6-0) = -6

$A_{13} = + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$ = (4+1) = 5

$A_{21} = - \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$ = - (3+2) = -5

$A_{22} = + \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}$ = (3-1) = 2

$A_{23} = - \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$ = - (2+1) = -3

$$A_{31} = +\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = (0+1) = 1$$

$$A_{32} = -\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -(0+2) = -2$$

$$A_{33} = +\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1-2) = -1$$

$$\text{Co-factor matrix } A = \begin{pmatrix} 3 & -6 & 5 \\ -5 & 2 & -3 \\ 1 & -2 & -1 \end{pmatrix}$$

$$\text{Adj } A = \begin{pmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{\text{Adj}A}{|A|} = \frac{-1}{8}\begin{pmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{pmatrix}$$

EXERCISE

PART-A

1. Find the Value of $\begin{vmatrix} x+y & x \\ y+4z & y \end{vmatrix}$

2. Find the value of the determinant $\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}$

3. Find the value of minor 5 in the determinant $\begin{vmatrix} 1 & 0 & -1 \\ 5 & 2 & 4 \\ 3 & -2 & 6 \end{vmatrix}$

4. Find the value of 'm' so that $\begin{vmatrix} 2 & -4 & 1 \\ 4 & -2 & -1 \\ 3 & 1 & m \end{vmatrix} = 0$

5. Find the value of 'x' if $\begin{vmatrix} 1 & -1 & 2 \\ 5 & 3 & x \\ 2 & 1 & 4 \end{vmatrix} = 0$

6. Evaluate without expanding $\begin{vmatrix} 18 & 40 & 58 \\ 16 & 36 & 52 \\ 12 & 28 & 40 \end{vmatrix}$

7. Show that $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$

8. Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

9. Show that $\begin{vmatrix} 1 & ab & bc+ca \\ 1 & bc & ca+ab \\ 1 & ca & ab+bc \end{vmatrix} =0$

10. Find the value of $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 2a & 2b & 2c \end{vmatrix}$

11. Show that $\begin{vmatrix} 4 & -1 & 2 \\ 2 & 3 & 8 \\ 1 & 2 & 5 \end{vmatrix} = 0$

12. If A = $\begin{pmatrix} 1 & 2 \\ 6 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ find 2A – B

13. Find 2x2 matrix A if $a_{ij} = i+j$

14. If f(x) = x+3 and A = $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ find f(A)

15. If f (x) = 2x-5 and A = $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ find f(A)

16. Show that the matrix $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 2 & -4 & 6 \end{pmatrix}$ is singular

17. Prove that the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is non-singular

18. If A = $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ find AB

19. If X = $\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ and Y = $\begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$ find XY.

20. If A = $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ find A^2

21. Find the co-factor matrix of $\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

22. Find the adjoint of $\begin{pmatrix} 2 & -1 \\ 6 & 1 \end{pmatrix}$

23. Find the in inverse of $\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$

24. Find the in inverse of $\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$

PART – B

1. Solve by Cramer's rule
   a. $3x - y + 2z = 8$, $x - y + z = 2$ and $2x + y - z = 1$
   b. $3x + y + z = 3$, $2x + 2y + 5z = -1$ and $x - 3y - 4z = 2$
   c. $x + y + z = 3$, $2x + 3y + 4z = 9$ and $3x - y + z = 3$
   d. $x + y + z = 3$, $2x - y + z = 2$, $3x + 2y - 2z = 3$
   e. $x + y - z = 4$, $3x - y + z = 4$, $2x - 7y + 3z = -6$
   f. $x + 2y - z = -3$, $3x + y + z = 4$, $x - y + 2z = 6$

2. Prove that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

3. Prove that $\begin{vmatrix} x+y & z & z \\ x & y+z & x \\ y & y & z+x \end{vmatrix} = 4xyz$

4. Prove that $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

5. Prove that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

6. If $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$ show that AB = BA

7. If $A = \begin{pmatrix} 3 & 6 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ 2 & 3 \end{pmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$

8. Find the inverse of the following

   i) $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$

   ii) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ -1 & -2 & -1 \end{pmatrix}$

   iii) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{pmatrix}$

   iv) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -3 \\ 6 & -2 & -1 \end{pmatrix}$

ANSWERS

PART - A

1 y^2-4xz

2 1

3 -1

4 M = -2

5 x = 10

6 0

10 0

12 $\begin{pmatrix} 1 & 2 \\ 13 & -2 \end{pmatrix}$

13 $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

14 $\begin{pmatrix} 5 & 3 \\ 1 & 4 \end{pmatrix}$

15 $\begin{pmatrix} -3 & -2 \\ 0 & -1 \end{pmatrix}$

18 $\begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}$

19 $\begin{pmatrix} 1 & -5 \\ -2 & 0 \end{pmatrix}$

20 $\begin{pmatrix} 5 & 6 \\ 6 & 17 \end{pmatrix}$

21 $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

22. $\begin{pmatrix} 1 & 1 \\ -6 & 2 \end{pmatrix}$

23. $\frac{1}{11}\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$

24. $\frac{1}{6}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

PART - B

1. a) 1,3,4

   b) 1,1,-1

   c) 1,1,1

   d) 1,1,1

   e) 2,1,-1

   f) 1,-1,2

8. i) $\frac{1}{20}\begin{pmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{pmatrix}$

   ii) $\frac{-1}{6}\begin{pmatrix} -3 & -1 & -1 \\ 0 & -2 & 4 \\ -3 & 3 & -3 \end{pmatrix}$

   iii) $\begin{pmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{pmatrix}$

   iv) $\frac{-1}{15}\begin{pmatrix} -9 & -3 & 0 \\ -16 & -7 & 5 \\ -22 & -4 & 5 \end{pmatrix}$

UNIT – II

BINOMIAL THEOREM

2.1 Definition of factorial notation, definition of Permutation and Combinations with formula. Binomial theorem for positive integral index (statement only), finding of general and middle terms. Simple problems.

2.2 Problems finding co-efficient of x^n, independent terms. Simple problems. Binomial Theorem for rational index, expansions, only upto – 3 for negative integers. Simple Expansions.

2.3 **Partial Fractions**

Definition of Polynomial fraction, proper and improper fractions and definition of partial fractions.

To resolve proper fraction into partial fraction with denominator containing non repeated linear factors, repeated linear factors and irreducible non repeated quadratic factors. Simple problems.

2.1 BINOMIAL THEOREM

Definition of Factorial Notation:

The continued product of first 'n' natural numbers is called "n factorial' and is denoted by n! or $\lfloor n$

ie $n! = 1.2.3.4\ldots\ldots.. (n-1).n$

$5! = 1.2.3.4.5 = 120$

Zero factorial: we will require zero factorial for calculating any value which contains zero factorial. It does not make any sense to define it as the product of the integers from 1 to zero. So we define 0!=1

Deduction: $n! = 1.2.3.4\ldots\ldots\ldots..(n-1)n$

$= [1.2.3.4\ldots\ldots..(n-1)]n$

$= (n-1)!n$

Thus, $n! = n[(n-1)!]$

For Example,

$9!= 9(8!)$

Binomial Theorem

Introduction:

Before introducing Binomial theorem, first we introduce some basic ideas and notation.

First we begin with the following problem. We want to select 2 carom players from among 5 good carrom players. Let us denote them by the letters A,B,C, D and E.

To Select 2 players, we shall first take A and then with him, associate B,C,D and E. That is AB, AC, AD and AE are four types of selection of 2 players. Also starting with B, we have BC, BD and BE with C, CD and CE and finally starting with D, we have DE only.

So totally there are 10 ways, for selecting 2 players out of 5 players, If we denote the number of ways of selection of 2 persons out of 5 persons, symbolically by $5C_2$, then we have $5C_2 = 10$

Also let us assume that the selected 2 players are going to Pune for national level competition. Since attending such national level competition itself is admirable, we want to give pose for a group photo. Two chairs are brought as shown below, there are two types of arrangements

Arrangement – 1		Arrangement – 2	
$\boxed{A}$	$\boxed{B}$	$\boxed{B}$	$\boxed{A}$
Chair 1	Chair 2	Chair 1	Chair 2

Thus we see that for one selection, there are two different arrangements and so for the total of 10 selection, the total number of arrangements is 20.

That is if the number of ways of arrangement of 2 persons out of 5 persons, is denoted by $5P_{2,}$ we have $5P_2 = 20$.

The above “arrangement” and “selection” are usually called “permutation and combination”. They mean the arrangement by the work permutation and the selection, by the work combination.

So the number of ways of arrangement and the number of ways of selection in the above example are respectively denoted by $5P_2$ and $5C_2$.

Hence we have $5P_2$ = 20 and $5C_2$ = 10, as seen in the above example of carom players, and the method of calculating $5P_2$ or $5C_2$ can be remembered as below:

$$5P_2 = 5(5-1) = 5x4 = 20$$

$$5C_2 = \frac{5(5-1)}{1x2} = \frac{5x4}{1x2} = 10$$

Examples:

1) $7P_3 = 7.6.5 = 7x30 = 210$

and $7C_3 = \frac{7.6.5}{1.2.3} = 35$

2) $11p_4 = 11.10.9.8 = 110 \times 72 = 7920$

And $11C_4 = \frac{11.10.9.8}{1.2.3.4} = 11 \times 30 = 330$

Note : Selection of 11 cricket players out of 17 players can be done in $17C_{11}$ ways and

$$17C_{11} = \frac{17.16.15.14.13.12(11.10.9.8.7)}{1.2.3.4.5.6.(7.8.9.10.11)}$$

$$= \frac{17.16.15.14.13.12}{1.2.3.4.5.6} = 17C_6$$

That is, selection of 11 out of 17 is same as selection of 6 out of 17. In general, $nc_r = nc_{n-r}$

Also it must be noted that the number of ways of selection or arrangement (combination or permutation) will always be a positive integer and it can never be a fraction.

Definition: np_r is the no.of ways of arrangement (or permutation) of r things out of n things.

nc_r is the no. of ways of selection (or combination) of 'r' things out of 'n' things.

(or)

np_r means number of permutation of n things, taken 'r' at a time and nc_r means the no.of ways combining 'n' things, taken 'r' at a time.

Note:

The values of np_r and n_{Cr} are given below

$$np_r = n(n-1)(n-2)....(n-r+1) = \frac{n!}{(n-r)!}$$

$$nc_r = \frac{n(n-1)(n-2)........(n-r+1)}{1.2.3......r} = \frac{n!}{(n-r)!r!}$$

where $n! = 1 \times 2 \times 3 \times ....\times n$

Examples:

$np_1 = n$, np_2, $= n(n-1)$, $np_3 = n(n-1)(n-2)$

$$nc_1 = \frac{n}{1},\ nc_2\ \frac{n(n-1)}{1.2},\ nc_3 = \frac{n(n-1)(n-2)}{1.2.3} \text{ etc.,}$$

2.2 BINOMIAL THEOREM

Binomial means an expression, which consists of two numbers or group of numbers connected by plus sign or minus sign

Example:

$$x + y,\ 2x - y,\ a + 2b,\ 3 - (a + b),\ x^3 - \frac{1}{x} \text{ etc.,}$$

In binomial theorem, we deal with the powers of binomial expressions. From school studies, we know that

$(x+a)^2 \quad = x^2 + 2xa + a^2$

and $(x+a)^3 \quad = x^3+3x^2.a + 3xa^2 + a^3$

After studying about the values of nc_r (nc_1, nc_2, nc_3,..........etc), we can understand and

write the expansion of $(x+a)^3$ as below

$(x+a)^3 = x^3 + 3c_1x^2.a + 3c_2x.a^2+a^3$

Similarly for $(x+a)^4$ and $(x+a)^5$, we can write as below:

$(x+a)^4 = x^4+4c_1.x^3a +4c_2.x^2a^2+4c_3x.a^3+a^4$ and

$(x+a)^5 = x^5+5c_1.x^4.a + 5c_2.x^3a^2+ 5c_3x^2.a^3+5c_4xa^4 +a^5$

Statement: Binomial theorem for a positive integral index

If n is any positive integer, then

$(x+a)^n = x^n+nc_1.x^{n-1}.a+nc_2.x^{n-2}.a^2+nc_3.x^{n-3}a^3+.......\ +nc_r.x^{n-r}.a^r+........+a^n.$

Notes :

1) The total number of terms in the expansion is (n+1)
2) In each term, sum of the powers (exponents) of x and a is equal to n.
3) The general term $ncr.x^{n-r}.a^r$ is $(r+1)^{th}$ term. Ie $t_{r+1} = nc_r.x^{n-r}.a^r$
4) nc_0,nc_1,nc_2........etc are called binomial co-efficient
5) Since $nc_r = nc_{n-r}$

We have $nc_0 = n_{cn}$, $nc_1=nc_{n-1}$, $nc_2=nc_{n-2,}$ etc.,

It must be noticed that $nc_0=nc_n=1$

6) If 'n' is an even integer, there is only one middle term which will be at $\left(\frac{n}{2}+1\right)^{th}$ place and if 'n' is odd number there are two middle terms, which are at $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ places.
7) To find the term independent of x in the binomial expansion, put the power of x in the general term as zero
8) In finding the term independent of x, if the value of r comes to be a fraction, then it means that other is no term independent of x.

Binomial Theorem for rational index: If x is numerically less than one and n, any rational number, then,

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+...........$$

Note :

1) The no. of tems in the expansion is infinite
2) Here the notations nc_0,nc_1,nc_2.......etc are meaningless; since n is a rational number
3) Also here we consider, only expansions of negative integers upto-3

When the values of 'n' are -1,-2,-3 the expansions are

1) $(1-x)^{-1} = 1+x+x^2+x^3$..............

2) $(1+x)^{-1} = 1-x+x^2-x^3$..............

3) $(1-x)^{-2} = 1+2x+3x^2+4x^3$..............

4) $(1+x)^{-2} = 1-2x+3x^2-4x^3$..............

5) $(1-x)^{-3} = 1+3x+6x^2+10x^3+15x^4$..............

6) $(1+x)^{-3} = 1-3x+6x^2-10x^3+15x^4$..............

2.1 WORKED EXAMPLES

PART – A

1.Expand $(2x-y)^4$ using binomial therorem

$(x+a)^n = x^n+nc_1.x^{n-1}.a + nc_2.x^{n-2}.a^2 + ...+nc_r.x^{n-r}.a^r+...+a^n$

$$(2x-y)^4 = [2x+(-y)]^4 = (2x)^4 + 4c_1(2x)^3(-y)+4c_2.(2x)^2 (-y)^2+4c_3.2x.(-y)^3 +(-y)^4$$

$$= 16x^4 - 32x^3.y+24x^2.y^2 - 8xy^3 + y^4$$

2.Expand $\left(x+\frac{1}{x}\right)^5$ using binomial theorem

$$\left(x+\frac{1}{x}\right)^5 = x^5 + 5c_1.x^4.\left(\frac{1}{x}\right) + 5c_2.x^3.\left(\frac{1}{x}\right)^2 + 5c_3.x^2\left(\frac{1}{x}\right)^3 + 5c_4.x.\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5$$

$$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$$

$$= \left(x^5 + \frac{1}{x^5}\right) + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$$

3.Find the general term in the expansion of $\left(x-\frac{1}{x}\right)^{10}$

Solution : The general term,

$$t_{r+1} = nc_r.x^{n-r}.a^r$$

$$= 10cr.(x)^{10-r}.\left(-\frac{1}{x}\right)^r \text{,since here n=10,} \quad x = x \text{ and } a = -\frac{1}{x}$$

$$= 10cr.x^{10-r}.(-1)^r.x^{-r} = 10c_r.(-1)^r.x^{10-2r}$$

4. Find the 5^{th} term in the expansion of $\left(x+\frac{1}{x}\right)^8$

Solution: The general term, $t_{r+1} = nc_r.x^{n-r}.a^r$

Here r+1 = 5

r = 4 we get 5^{th} term t_5

Also n=8, x=x and a = $\frac{1}{x}$

$$t_5 = 8c_4.x^{8-4}.\left(\frac{1}{x}\right)^4$$

$$= 8c_4.x^4.\frac{1}{x^4} = 8c_4 = \frac{8.7.6.5}{1.2.3.4} = 70$$

PART – B

1.Find the middle term in the expansion of $\left(x+\frac{1}{x}\right)^8$

Solution: Here n is even number and so there is only one middle term and that is $\left(\frac{8}{2}+1\right)^{th}$ term.

So, 5^{th} term is the middle term.

Now, general term is $t_{r+1} = nc_r.x^{n-r}.a^r$

To find t_5, put r =4

$$\therefore t_5 = t_{4+1} = 8c_4.(x)^{8-4}\left(\frac{1}{x}\right)^4$$

$$= \frac{8.7.6.5}{1.2.3.4}x^4.\frac{1}{x^4}$$

= 14 x 5 = 70

2.Find the middle terms in the expansion of $\left(2x^2 + \frac{1}{x}\right)^{11}$

Solution:

Since here n is odd number, the total no. of terms in the expansion is even and so there are two middle terms $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms are the middle terms. i.e, $\left(\frac{11+1}{2}\right)^{th}$ and $\left(\frac{11+3}{2}\right)^{th}$ terms, that is 6^{th} and 7^{th} term are the middle terms

Now, general term is $t_{r+1} = nc_r.x^{n-r}\ a^r$

$$t_6 = t_{5+1} = 11c_5.(2x^2)^{11-5}.\left(\frac{1}{X}\right)^5$$

$$= 11c_5\ 2^6.(x^2)^6.\ 1/x^5$$

$$= 11c_5.2^6.(x^{12}).\ 1/x^5$$

$$= 11c_5\ 2^6.x^{12-5} = 11c_5\ 2^6.x^7$$

$$t_7 = t_{6+1} = 11c_6(2x^2)^{11\text{-}6}\left(\frac{1}{x}\right)^6$$

$$= 11c_6\left(2x^2\right)^5\left(\frac{1}{x}\right)^6$$

$$= 11c_6.2^5x^{10}.\frac{1}{x^6}$$

$$= 11c_6\,2^5x^4.$$

3.By using Binomial theorem, find the 6^{th} power of 11

Solution :

$$11^6 = (10+1)^6$$

$$= 10^6 + 6c_1.10^5 + 6c_2.10^4.1^2 + 6c_3.10^3.1^3 + 6c_4.10^2.1^4$$
$$+ 6c_5.10.1^5 + (1)^6$$

$$= 10^6 + \frac{6}{1}.10^5 + \frac{6.5}{1.2}.10^4 + \frac{6.5.4}{1.2.3}.10^3 + 6c_2 10^2 + 6c_1 10 + 1^6$$

$$= 10^6 + 6.10^5 + 15.10^4 + 20.10^3 + 15.10^2 + 6.10 + 1$$

$$= \left\{ \begin{array}{r} 10,00,000 \\ 6,00,000 \\ 1,50,000 \\ 20,000 \\ 1,500 \\ 60 \\ 1 \end{array} \right\} = 1,771,561$$

4.Using Binomial theorem, find the value of $(1.01)^5$ correct to 3 decimal places.

Solution :

$$(1.01)^5 = (1+0.1)^5$$

$$= 1^5 + 5c_1.1^4(0.01) + 5c_2.1^3.(0.01)^2 + 5c_3.1^2.(0.01)^3$$
$$+ 5c_4.1^1.(0.01)^4 + (0.01)^5$$

$$= 1 + \frac{5}{1}(0.01) + \frac{5.4}{1.2}(0.01)^2 + 5c_2(0.01)^3 + \ldots\ldots\ldots\ldots$$

$$= 1 + 0.05 + 10.(0.0001) + \ldots\ldots\ldots\ldots$$

$$= 1 + 0.05 + 0.001 + \ldots\ldots\ldots\ldots = 1.051,$$

Correct to 3 decimal places.

2.2 WORKED EXAMPLES

PART – A

1. Find the expansion of $(1+x)^{-2}$

Solution:

We know that $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{1.2} + \frac{n(n-1)(n-2)x^3}{1.2.3} + ...$

$$\therefore (1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-2-1)}{1.2}(x^2) + \frac{(-2)(-2-1)(-2-2)x^3}{1.2.3} + ...$$

$$= 1 - 2X + \frac{(-2)(-3)x^2}{1.2} + \frac{(-2)(-3)(-4)}{1.2.3}x^3$$

$$= 1 - 2x + 3x^2 - 4x^3 + ...$$

2. Find the expansion of $(1-x)^{-3}$

Solution:

$$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-3-1)(-x)^2}{1.2} + \frac{(-3)(-3-1)(-3-2)}{1.2.3}$$

$$(-x)^3 + ...$$

$$= 1 + 3x + 6x^2 + 10x^3 + ...$$

PART – B

1.Find the co-efficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Solution:

Now, general term is $t_{r+1} = nc_r.x^{n-r}.a^r$

$$t_{r+1} = 15c_r.(x^4)^{15-r}.\left(\frac{-1}{x^3}\right)^r$$

$$= 15c_{r.} x^{60-4r} (-1)^r x^{-3r}$$

$$= 15c_r.(-1)^r.x^{60-7r} \quad (1)$$

To find the co-efficient of x^{32}, put 32 = 60 – 7r

$$\therefore 7r = 60 - 32 = 28 \text{ and } r = \frac{28}{7} = 4$$

Applying r = 4 in the equation (1), we get

$$t_{4+1} = 15c_4.(-1)^4.x^{60-28}$$

i.e., $t_5 = 15c_4.1.x^{32}$. so, co-efficient of x^{32} is $15c_4$.

2. Find the term independent of x in the expansion of $\left(x - \frac{1}{x}\right)^{10}$

Solution:

Now, general term is $t_{r+1} = nc_r.x^{n-r}.a^r$

$$t_{r+1} = 10c_r.(x)^{10-r}.\left(-\frac{1}{x}\right)^r$$

$$= 10c_r.(x)^{10-r}.(-1)^r.x^{-r}$$

$$= 10c_r.(-1)^r x^{10-2r} \quad (1)$$

To find independent term, put 0 =10-2r $\therefore$ 2r = 10

and r =5

Using the value of r=5 in equation (1), we get the independent term as

$$t_{5+1} = 10c_5.(-1)^5 x^{10-10}$$

$$= 10c_5.(-1)x^0$$

$$= -10c_5 = \frac{-10.9.8.7.6}{1.2.3.4.5} = -63 \text{ x } 4$$

$$= -252$$

2.Expand $(3-4x)^{-3}$ using Binomial theorem

Solution:

$$(3-4x)^{-3} = 3^{-3}\left[1-\tfrac{4}{3}.x\right]^{-3}$$

$$=\frac{1}{3^3}\left[1+(-3)\left(\frac{-4}{3}x\right)+\frac{(-3)(-4)}{1.2}\left(\frac{-4}{3}x\right)^2+\frac{(-3)(-4)(-5)}{1.2.3}\left(\frac{-4}{3}x\right)^3+.....\right]$$

$$=\frac{1}{27}\left[1+4x+6\frac{(16x^2)}{9}+10\left(\frac{64}{27}\right)x^3+..........\right]$$

$$=\frac{1}{27}\left[1+4x+\frac{32}{3}x^2+\frac{640}{27}x^3+..........\right]$$

2.3 PARTIAL FRACTIONS

Definition of Polynomial Fraction:

An expression of the form $\frac{p(x)}{q(x)}$ where p(x) and q(x) ≠ o are polynomials in x is called a polynomial fraction.

The expressions $\frac{5x-2}{x^2+3x+2}$, $\frac{3x^2+2x-1}{x^2+x-22}$ are examples for rational or polynomial fraction.

Proper Fraction; A proper fraction is one in which the degree of the numerator is less than degree of the denominator.

The expressions $\frac{3x+1}{x^2+4x+3}$, $\frac{7x^2+9}{x^3+x^2-5}$ are examples for proper fraction.

Improper fraction:

An improper fraction is a fraction in which the degree of the numerator is greater than or equal to the degree of the denominator

The expressions $\frac{x^3+5x^2+4}{x^2+2x+3}$, $\frac{x^2-x+1}{x^2+x+3}$ are examples for improper fractions.

Partial Fraction

Consider the sum of $\frac{5}{x+2}$ and $\frac{3}{x+1}$

We simplify it as follows:

$$\frac{5}{x+2}+\frac{3}{x+1}=\frac{5(x+1)+3(x+2)}{(x+2)(x+1)}=\frac{5x+5+3x+6}{(x+2)(x+1)}$$

$$=\frac{8x+11}{(x+2)(x+1)}$$

Conversely the process of writing the given fraction $\frac{8x+11}{(x+2)(x+1)}$ as $\frac{5}{x+2}+\frac{3}{x+1}$ is known as splitting into partial fractions or expressing as partial fraction.

A given proper fraction can be expressed as the sum of other simple fractions corresponding to the factors of the denominator of the given proper fraction. This process is called splitting into Partial Fraction. If the given fraction $\frac{p(x)}{q(x)}$ is improper then convert into a sum of a polynomial expression and a proper rational fraction by dividing p(x) by q(x).

Working rule:

Given the proper fraction $\frac{p(x)}{q(x)}$. Factorise q(x) into prime factors.

Type 1

To resolve proper fraction into partial fraction with denominator containing non-repeated linear factors.

If ax+b is a linear factor of the denominator q(x), then corresponding to this factor associate a simple factor $\frac{A}{ax+b}$, where A is a constant (A≠0) ie., when the factors of the denominator of the given fraction are all linear factors none of which is repeated, we write the partial fraction as follows.

$\frac{x+2}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$ where A and B are constants to be determined.

Type 2: Repeated linear factors

If a linear factor ax+b occurs n times as factors of the denominator of the given fraction, then corresponding to these factors associate the sum of n simple fractions,

$$\frac{A1}{ax+b} + \frac{A2}{(ax+b)^2} + \frac{A3}{(ax+b)^3} + \dots\dots\dots\dots \frac{An}{(ax+b)^n}$$

Where A1, A2, A3,…………….An are constants.

Type 3 Irreducible non repeated quadratic factors

If a quadratic factor ax^2+bx+c which is not factorable into linear factors occurs only once as a factor of the denominator of the gives fraction, then corresponding to this factor $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants which are not both zeros.

Consider $\frac{3x}{(x-1)(x^2+1)}$

We can write this proper fraction in the form

$$\frac{3x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

The first factor of the denominator (x-1) is of first degree, so we assume its numerator as a constant A. The second factor of the denominator x^2+1 is of 2[nd] degree and which is not reducible into linear factors. We assume its numerator as a general first-degree expression Bx+c.

2.3 WORKED EXAMPLES

PART – A

1.Split up $\frac{x+2}{x(x+3)}$ into partial fraction without finding the constant

Solution:

$$\frac{x+2}{x(x+3)}=\frac{A}{x}+\frac{B}{x+3}$$

2.Without finding the constants split $\frac{x+4}{(x^2-4)(x+1)}$

Solution:

$$\frac{x+4}{(x^2-4)(x+1)}=\frac{x+4}{(x+2)(x-2)(x+1)}=\frac{A}{x+2}+\frac{B}{x-2}+\frac{C}{x+1}$$ where A, B and C and are constants.

3.Split $\frac{x+1}{x^2-5x+6}$ without finding the constants

Solution:

$$\frac{x+1}{x^2-5x+6}=\frac{x+1}{(x-3)(x-2)}=\frac{A}{x-3}+\frac{B}{x-2}$$ where A and B are constants

4.Without finding the constants split $\frac{5}{(x+1)(x-2)^2}$ into partial fraction.

Solution:

$$\frac{5}{(x+1)(x-2)^2}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

5.Split $\frac{4x}{(x-1)(x^2+1)}$ into partial fraction without finding the constants.

Solution:

$$\frac{4x}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$ where A, B and C are constants.

PART – B

1. Resolve $\frac{x+3}{(x+5)(2x+1)}$ into a partial fraction

Solution:

$$\text{Let } \frac{x+3}{(x+5)(2x+1)} = \frac{A}{x+5} + \frac{B}{2x+1}$$

$$\Rightarrow \frac{x+3}{(x+5)(2x+1)} = \frac{A(2x+1)+B(x+5)}{(x+5)(2x+1)}$$

$$\Rightarrow x+3 = A(2x+1)+B(x+5) \qquad (1)$$

Equating the co-efficient of the like powers of x,

We get,

Co-efficient's of x : 1 = 2A + B (2)

Constant term: 3 = A + 5B (3)

Solving (2) and (3) we get

$$A = \frac{2}{9} \text{ and } B = \frac{5}{9}$$

$$\therefore \frac{x+3}{(x+5)(2x+1)} = \frac{\frac{2}{9}}{x+5} + \frac{\frac{5}{9}}{2x+1}$$

Note: The constants A and B can also be found by successively giving suitable values of x.

2.Resolve $\frac{x-2}{(x+2)(x-1)^2}$ into a partial fraction.

Solution:

$$\text{Let } \frac{x-2}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\Rightarrow \frac{x-2}{(x+2)(x-1)^2} = \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)^2}$$

$$\Rightarrow x-2 = A(x-1)^2 + B(x+2)(x-1) + C(x+2) \qquad (1)$$

To find C, put x=1 in (1)

$$1-2 = A(1-1)^2 + B(1+2)(1-1) + C(1+2)$$
$$-1 = 0 + 3B \text{ X } 0 + 3C$$
$$C = -1/3$$

To find A put x= -2 in (1)

$$-2-2 = A(-2-1)^2 + B(-2+2)(-2-1) + C(-2+2)$$
$$-4 = A(-3)^2 + 0 + 0$$
$$-4 = 9A$$
$$A = \frac{-4}{9}$$

To find B, equating co-efficient of x^2 on both sides,

$$0 = A + B$$
$$0 = \frac{-4}{9} + B$$
$$B = \frac{4}{9}$$

$$\therefore \frac{x-2}{(x+2)(x-1)^2} = \frac{\frac{-4}{9}}{x+2} + \frac{\frac{4}{9}}{x-2} - \frac{\frac{-1}{3}}{(x+1)^2}$$

3.Resolve : $\frac{2x+1}{(x+1)(x^2+1)}$ into a partial fraction

Solution:

Let $\frac{2x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$$\frac{(2x+1)}{(x+1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$\Rightarrow 2x+1 = A(x^2+1)+(Bx+C)(x+1) \qquad (1)$$

To Find A, put x = -1 in (1)

$$-2+1 = A[1+1]+(-B+C)(0)$$

$$-1 = 2A$$

$$A = -1/2$$

To find C, put x = 0 in (1)

$$2x0+1 = A(0^2+1)+(Bx0+C)(0+1)$$

$$1 = A+C$$

$$1 = -1/2+C$$

$$C = 1+1/2 = \frac{2+1}{2} = \frac{3}{2}$$

To find B, put x=1 in (1)

$$2x1+1 = A(1^2+1)+(Bx1+c)(1+1)$$

$$2+1 = A(2)+(B+C).2$$

$$3 = 2A+2B+2C$$

$$3 = 2\left(-\frac{1}{2}\right)+2B+2\left(\frac{3}{2}\right)$$

$$3 = -1+2B+3$$

$$3 = 2+2B$$

$$2B = 3-2 = 1$$

$$B = \frac{1}{2}$$

$$\therefore \frac{2x+1}{(x+1)(x^2+1)} = \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x+\frac{3}{2}}{x^2+1} = \frac{\frac{1}{2}x+\frac{3}{2}}{x^2+1} - \frac{\frac{1}{2}}{x+1}$$

EXERCISE

PART – A

1.State Binomial theorem for positive integral index

2.Expand $[2x+3y]^3$ using binomial theorem.

3.Expand $[a-2b]^5$ using binomial theorem.

4.Expand $(3x-2y)^3$ using binomial theorem.

5.Expand $[5x-y]^4$ using binomial theorem

6.Find the general term in the expansion of $\left(x^2 - \frac{2}{x}\right)^9$

7.Find the general term in the expansion of $\left(x - \frac{1}{x}\right)^{10}$

8.Find the general term in the expansion of $\left(x + \frac{2}{x}\right)^{12}$

9.Find the general term in the expansion of $\left(x^2 + \frac{1}{x^2}\right)^8$

10. Find the general term in the expansion of $\left(4x^3 + \frac{1}{x^2}\right)^{11}$

11. Expand $(1-2x)^{-3}$ using binomial theorem.

12. Write the first 3 terms of $(1+x)^{-3}$

13. Write the first three terms of $(1-2x)^{-3}$

14. Expand $(1+x)^{-1/3}$ upto 3 terms

15. Expand $(1-x^2)^{-2}$ binomially

16. Write the first 3 terms of $(1-4x)^{-2}$

17. Split $\frac{x-1}{x(x-1)}$ into partial fraction without finding the constants.

18. Split $\dfrac{x+2}{x(x^2-1)}$ into partial fraction without finding the constants.

19. Split $\dfrac{2x-1}{(x+1)(x+2)^2}$ into partial fraction without finding the constants.

20. Split $\dfrac{x^2-3}{(x+2)(x^2+1)}$ into partial fraction without finding the constants.

PART – B

1.Find the middle term in the expansion of $\left(x-\dfrac{2}{x}\right)^{12}$

2.Find the middle term in the expansion of $\left(5x-\dfrac{2}{3x^2}\right)^{10}$

3.Find the 16^{th} term in the expansion of $\left(x-\dfrac{1}{x}\right)^{30}$

4.Find the 5^{th} term in the expansion of $\left(2x^2-\dfrac{3}{x}\right)^{11}$

5.Find the middle terms in the expansion of $\left(x+\dfrac{1}{x}\right)^{27}$

6.Find the middle terms in the expansion of $\left(2x-\dfrac{3}{x}\right)^{15}$

7.Find the middle terms in the expansion of $\left(4x^3+\dfrac{1}{x^2}\right)^{11}$

8.Find the co-efficient of x^{-5} in the expansion of $\left(x-\dfrac{3}{5x^2}\right)^{7}$

9.Find the co-efficient of x^{-17} in the expansion of $\left(2x^4 - \frac{3}{x^3}\right)^{15}$

10. Find the co-efficient of x^{11} in the expansion of $\left(x + \frac{2}{x^2}\right)^{17}$

11. Find the term independent of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^{12}$

12. Find the term independent of x in the expansion of $\left(x + \frac{1}{x}\right)^{10}$

13. Find the term independent of x in the expansion of $\left(\sqrt{x} - \frac{2}{3x^2}\right)^{10}$

14. Find the constant term in the expansion of $\left(2x + \frac{1}{x}\right)^{8}$

15. Using binomial theorem, find the value of 99^3

16. Resolve $\frac{7x-1}{6-5x+x^2}$ into a partial fraction

17. Resolve $\frac{x^2+x+1}{(x-1)(x-2)(x-3)}$ into a partial fraction

18. Resolve $\frac{1}{(x+1)(x+2)}$ into a partial fraction

19. Resolve $\frac{1}{(x-1)(x+2)^2}$ into a partial fraction.

20. Resolve $\frac{x-2}{(x+2)(x-3)^2}$ into a partial fraction.

21. Resolve $\frac{x+1}{(x-2)^2(x+3)}$ into a partial fraction.

22. Resolve $\dfrac{x^2-6x+2}{x^2(x+2)}$ into a partial fraction.

23. Resolve $\dfrac{x+2}{(x+1)(x^2+1)}$ into a partial fraction.

24. Resolve $\dfrac{1}{(x+1)(x+2)^2}$ into a partial fraction.

ANSWERS

PART – A

1.Refer statement

2. $(2x)^3+3c_1.(2x)^2.(3y)+3c_2.(2x)^1.(3y)^2+3c_3(2x)^0(3y)^3$

3. $a^5+5c_1.(a)^4.(-2b^1)+5c_2.a^3.(-2b)^2+5c_3.a^2(-2b)^3$
$+5c_4.a(-2b)^4+5c_5.(-2b)^5$

4. $(3x)^3+3c_1.(3x)^2(-2y)+3c_2.(3x)(-2y)^2+(-2y)^3$

5. $(5x)^4+4c_1.(5x)^3.(-y)^1+4c_2.(5x)^2.(-y)^2+4c_3.(5x)^1.(-y)^3+4c_4(-y)^4$

6. $T_{r+1}=9c_r.(-2)^r.x^{18-3r}$

7. $T_{r+1}=10c_r..x^{10-2r}.(-1)^r$

8. $T_{r+1}=12c_r.(2)^r.x^{12-2r}$

9 $8c_r\ x^{16-4r}$

10 $T_{r+1}=11c_r.4^{11-r}.1^r.x^{33-5r}$

11 $1+(-3).(-2x)+\dfrac{(-3)(-3-1)}{1x2}(-2x)^2+.............$

12 $1+(-3)x+\dfrac{(-3)(-3-1)}{1.2}x^2+.............$

13 $(1-2x)^{-3}=1+\frac{(-3)(-2x)}{1!}+\frac{(-3)(-3-1)(-2x)^2}{1.2}+...........$

14 $1+(\frac{-1}{3}x)+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)x^2}{1.2}+.................$

15 $1+\frac{(-2)(-x^2)}{1!}+\frac{(-2)(-2-1)}{1.2}(-x^2)^2+..............$

16 $1-\frac{2(-4x)}{1!}+\frac{(-2)(-2-1)}{1.2}(-4x)^2+.....................$

17 $\frac{A}{x}+\frac{B}{x-1}$

18 $\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

19 $\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$

20 $\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$

PART – B

1. $2^6.12c_6$

2. $10c_5\left(\frac{10}{3x}\right)^5$

3. $-30c_{15}$

4. $11c_4.2^7.3^4.x^{10}$

5. $27c_{13}.x;\ \frac{27c_{14}}{x}$

6. $2^8(-3)^7 15c_7 x$ and $2^7.3^8.15c_8.\frac{1}{x}$

7. $11c_5 4^6.x^8; 11c_6 4^5.x^3$

8. $\frac{567}{125}$

9. $-15c_4.2^4.3^{11}$

10. 544

11. $16.12c_8$

12. $10c_5$

13. $\frac{4}{9}.10c_2$

14. $2^4.8c_4$

15. 9, 70, 299

16. $\frac{20}{x-2} - \frac{13}{x-3}$

17. $\frac{\frac{3}{2}}{x-1} + \frac{(-7)}{x-2} + \frac{\frac{13}{2}}{x-3}$

18. $\frac{1}{x+1} - \frac{1}{x+2}$

19. $\frac{\frac{1}{9}}{x-1} - \frac{\frac{1}{9}}{x+2} - \frac{-1/3}{(x+2)^2}$

20. $\frac{\frac{-7}{25}}{x+2} - \frac{\frac{-1}{50}}{x-3} + \frac{\frac{1}{5}}{(x-3)^2}$

21. $\dfrac{\frac{-2}{25}}{x+3}+\dfrac{\frac{2}{25}}{x-2}+\dfrac{\frac{3}{5}}{(x-2)^2}$

22. $\dfrac{\frac{9}{2}}{x+2}+\dfrac{\left(\frac{-7}{2}+1\right)}{x^2}$

23. $\dfrac{\frac{1}{2}}{x+1}+\dfrac{\left(-1/2\,x+\frac{3}{2}\right)}{x^2+1}$

24. $\dfrac{1}{x+1}-\dfrac{1}{x+2}-\dfrac{1}{(x+2)^2}$

UNIT – III

STRAIGHT LINES

3.1 Length of perpendicular distance from a point to the line and perpendicular distance between parallel lines. Simple problems.

Angle between two straight lines and condition for parallel and perpendicular lines. Simple problems

3.2 Pair of straight lines Through origin

Pair of lines passing through the origin $ax^2+2hxy+by^2=0$ expressed in the form $(y-m_1x)(y-m_2x)=0$. Derivation of

$\tan\theta = +\dfrac{2\sqrt{h^2-ab}}{a+b}$ condition for parallel and perpendicular lines. Simple problems.

3.3 Pair of straight lines not through origin

Condition for general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent pair of lines.

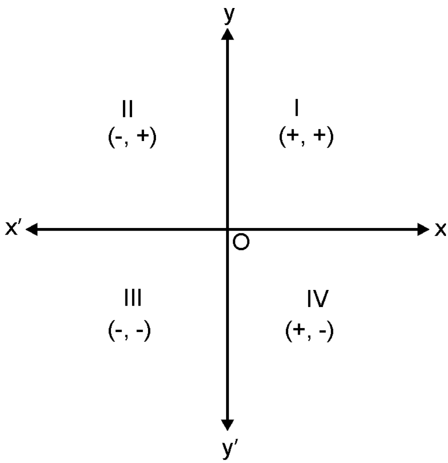
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ (Statement only)}$$

Angle between them, condition for parallel and perpendicular lines simple problems.

STRAIGHT LINES

Introduction

Analytical Geometry is a branch of Mathematics which deals with solutions of geometrical problems by Algebraic methods. It was developed by the famous French mathematician called Rune Descartes.

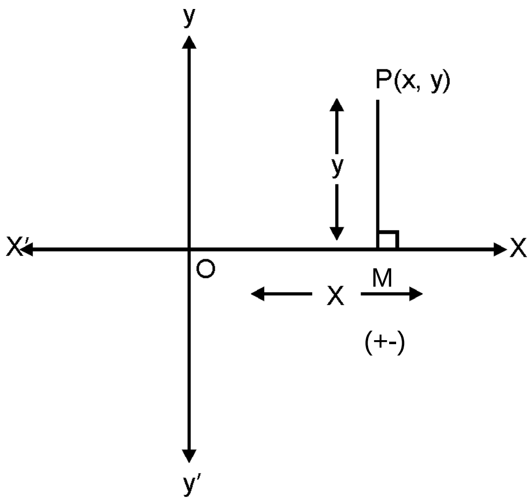


Axes of co-ordinates:

Take two straight lines XOX' and YOY' at right angles to each other. The horizontal line XOX' is called the X – axis and the vertical line YOY' is called the Y-axis. These two axes intersect at O, called the origin.

Cartesian – Rectangular Co-ordinates:

Diagram



Let XOX' and YOY' be the axes of co-ordinates. Let P be any point in the plane. Draw PM perpendicular to OX. Then the position of P is uniquely determined by the distances OM and MP. These distances OM and MP are called the Cartesian rectangular co-ordinates of the point P with respect to X-axis and Y–axis respectively.

It is to be noted that the 'X' co-ordinate must be in first place and the 'Y' co-ordinate must be in the second place. This order must be strictly followed.

Straight Line:

When a variable point moves in accordance with a geometrical law, the point will trace some curve. This curve is known as the locus of the variable point.

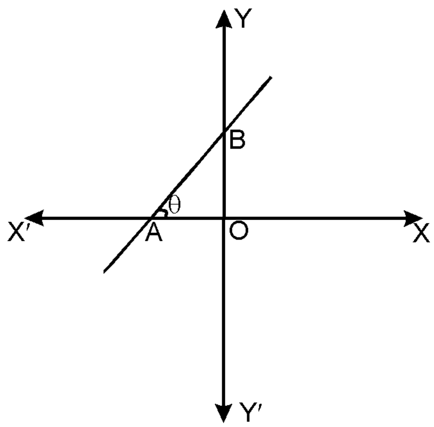
If a relation in x and y represent a curve then

(i) The co-ordinates of every point on the curve will satisfy the relation.

(ii) Any point whose co-ordinates satisfy the relation will lie on the curve.

Straight line is a locus of a point.

Diagram



Let the line AB cut the X-axis at A and y-axis at B. The angle made by the line AB with the positive direction of the x-axis is called the angle of inclination of the line AB with the x-axis and it is denoted by θ. Hence $\angle XAB = \theta$.The angle can take any values from 0° to 180°.

Slope or gradient of a straight line:

The tangent of the angle of inclination of the straight line is called slope or gradient of the line. If θ is the angle of inclination then

slope = $\tan\theta$ and is denoted by m.

Example:

If a line makes an angle of 45° with the X-axis in the positive direction then the slope of the line is $\tan 45^\circ$.

i.e $m = \tan 45^\circ = 1$

In school studies students have learnt, the distance between two points section formula, mid point of the line joining two points, various form of equation of the straight line, point of intersection of two lines, etc., in analytical Geometry.

Standard forms of the equation of a straight line.

(i) **Slope – intercept form:**

When 'c' is the y intercept and slope is 'm', the equation of the straight line is $y=mx+c$

(ii) **Slope – point form:**

When 'm' is the slope of the straight line and (x_1,y_1) is a point on the straight line its equation is $y-y_1 = m(x-x_1)$

(iii) **Two – point form:**

Equation of the line joining the two points (x_1,y_1) and (x_2,y_2) is

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

(iv) **Intercept form:**

When the x and y intercepts of a straight line are given as 'a' and 'b' respectively, the equation of the straight line is

ie., $\frac{x}{a}+\frac{y}{b}=1$

(v). **General form:**

The general form of the equation of a straight line is ax+by+c = 0. If ax+by+c=0 is the equation of a straight line then

$$\text{Slope } m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{a}{b}$$

$$x\text{-intercept} = -\frac{\text{constant term}}{\text{coefficient of } x} = -\frac{c}{a}$$

$$y\text{-intercept} = -\frac{\text{constant term}}{\text{coefficient of } y} = -\frac{c}{b}$$

Some Important Formulae:

(i) The length of the perpendicular from (x_1,y_1) to the line $ax + by + c = 0$ is

$$\pm\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$

(ii) The length of the perpendicular from origin to the line $ax + by + c = 0$ is

$$\pm\frac{c}{\sqrt{a^2+b^2}}$$

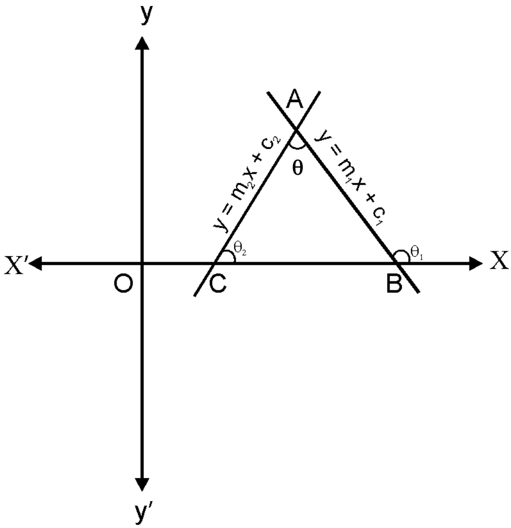
(iii) The distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is

$$\pm\frac{c_1-c_2}{\sqrt{a^2+b^2}}$$

ANGLE BETWEEN TWO STRAIGHT LINES

Book Work:

Find the angle between the lines $y = m_1x+c_1$ and $y = m_2x+c_2$. Deduce the conditions for the lines to be (i) parallel (ii) perpendicular



Proof:

Let θ_1 be the angle of inclination of the line $y = m_1x + c_1$. Slope of this line is $m_1 = \tan\theta_1$. Let 'θ'$_2$ be the angle of inclination of the line

$y = m_2x + c_2$. Slope of this line is $m_2 = \tan\theta_2$.

Let 'θ' be the angle between the two lines, then $\theta_1 = \theta_2 + \theta \Rightarrow \theta=\theta_1-\theta_2$

$$\therefore \tan\theta = \tan(\theta_1-\theta_2)$$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

$$\tan\theta = \frac{m_1-m_2}{1+m_1m_2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$$

(i) **Condition for two lines to be parallel:**

If the two lines are parallel then the angle between the two lines is zero

$$\therefore \tan\theta = \tan 0 = 0$$

(i.e) $$\frac{m_1 - m_2}{1 + m_1 m_2} = 0$$

$$m_1 - m_2 = 0$$

$$\therefore m_1 = m_2$$

$\therefore$ For parallel lines, slopes are equal.

(ii) **Condition for two lines to be perpendicular:**

If the two lines are perpendicular then the angle between them

$\theta = 90°$

$$\therefore \tan\theta = \tan 90° = \infty = \frac{1}{0}$$

$$\therefore \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{1}{0}$$

$$1 + m_1 m_2 = 0$$

$$m_1 m_2 = -1$$

$\therefore$ For perpendicular lines, product of the slopes will be -1

Note :

1) The acute angle between the lines

$$Y = m_1 x + c_1 \text{ and } y = m_2 x + c_2 \text{ is } \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

2) If the slope of a line is 'm' then the slope of parallel line is also m.

3) If the slope of a line is m then the slope of any line perpendicular to the line is $-\frac{1}{m}$

4) Any line parallel to the line ax+by+c = 0 will be of the form ax+by+d = 0 (differ only constant term)

5) Any line perpendicular the line ax+by+c =0 will be of the form bx-ay+d=0

3.1 WORKED EXAMPLES

PART – A

1) Find the perpendicular distance from the point (2,3) to the straight line 2x+y+3=0

Solution:

The length of the perpendicular from the point (x_1,y_1) to the line ax+by+c = is

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

Give straight line is 2x+y+3 =0

Given point (x_1,y_1) = (2,3)

(i.e) $\frac{2(2)+(3)+3}{\sqrt{(2)^2+(1)^2}} = \frac{10}{\sqrt{5}}$

2) Find the length of the perpendicular to the line 4x+6y+7 = 0 from the origin

Solution:

The length of the perpendicular is $\frac{c}{\sqrt{a^2+b^2}}$

Here a = 4, b=6, c =7

(i.e.) $\frac{7}{\sqrt{(4)^2+(6)^2}} = \frac{7}{\sqrt{16+36}} = \frac{7}{\sqrt{52}}$

3) Find the distance between the line 2x+3y+4 = 0 and 2x+3y -1 = 0

Solution:

The distance between the parallel lines is $\frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$

Here c_1 = 4 and c_2 = -1

Now $\frac{|4+1|}{\sqrt{(2)^2+(-3)^2}}$

$= \frac{5}{\sqrt{13}}$

4) Find the angle between the lines y = $\sqrt{3}$x and x-y = 0

Solution:

y = $\sqrt{3}x$

(i.e.) $\sqrt{3}x - y = 0$ (1) and x-y =0 (2)

Slope of (1) = $-\frac{\text{coefficient of x}}{\text{coefficient of y}}$

$= \frac{-\sqrt{3}}{-1} = \sqrt{3}$

tan θ_1 = $\sqrt{3} \Rightarrow \theta_1$ = 60°

slope of (2) = $-\frac{\text{coefficient of x}}{\text{coefficient of y}}$

$\tan\theta_2 = -\frac{1}{-1} = 1$

θ_2 = 45°

Let θ be the angle between (1) and (2)

∴ θ = θ_1 -θ_2

θ = 60° - 45° = 15°

5) Show that the lines 6x+y-11 =0 and 12x+2y+14 =0 are parallel

Solution:

$$6x+y-11 = 0 \qquad (1)$$

$$12x+2y+14 = 0 \qquad (2)$$

Slope of the line (1) = $m_1 = -\frac{a}{b} = -\frac{6}{1} = -6$

Slope of the line (2) = $m_2 = -\frac{a}{b} = -\frac{12}{2} = -6$

$m_1 = m_2$

$\therefore$ The lines are parallel.

6) Find 'p' such that the lines 7x-4y+13 = 0 and px = 4y+6 are parallel.

Solution:

$$7x-4y+13=0 \qquad (1)$$

$$px-4y-6 = 0 \qquad (2)$$

Slope of the line (1) $m_1 = \frac{-a}{b} = \frac{-7}{-4} = \frac{7}{4}$

Slope of the line (2) $m_2 = \frac{-a}{b} = \frac{-p}{-4} = \frac{p}{4}$

Since (1) and (2) are parallel lines

$m_1 = m_2$

$$\frac{7}{4} = \frac{p}{4}$$

$4p = 28$

$$p = \frac{28}{4} = 7$$

$\therefore$ p = 7

7) Show that the lines 2x+3y-7 =0 and 3x-2y+4=0 are perpendicular.

Solution:

$2x+3y-7 = 0 \quad (1)$

$3x-2y+4 = 0 \quad (2)$

Slope of the line (1) $= m_1 = \dfrac{-2}{3}$

Slope of the line (2) $= m_2 = \dfrac{-3}{-2} = \dfrac{3}{2}$

Now $m_1 m_2 = \left(\dfrac{-2}{3}\right)\left(\dfrac{3}{2}\right) = -1$

$\therefore m_1 m_2 = -1$

$\therefore$ The lines (1) and (2) are perpendicular

8) Find the value of m if the lines 2x+my=4 and x+5y-6=0 are perpendicular

Solution:

$2x+my-4=0 \quad (1)$

$x+5y-6 = 0 \quad (2)$

Slope of the line (1) $= m_1 = -\dfrac{2}{m}$

Slope of the line (2) $= m_2 = -\dfrac{1}{5}$

Since the liner are perpendicular

$m_1 \, m_2 = -1$

$$\left(-\frac{2}{m}\right)\left(-\frac{1}{5}\right) = -1$$

$$\frac{2}{5m} = -1$$

$-5m = 2$

$$m = -\frac{2}{5}$$

PART – B

1) Find the angle between the lines 3x+6y=8 and 2x = -y+5

Solution:

$3x+6y-8 = 0$ (1)

Slope of the line (1) $= m_1 = \frac{-a}{b} = \frac{-3}{6} = \frac{-1}{2}$

$2x+y-5 = 0$ (2)

Slope of the line (2) $= m_2 = \frac{-a}{b} = \frac{-2}{1} = -2$

Let 'θ' be the angle between two lines

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{-\frac{1}{2}+2}{1+\left(\frac{-1}{2}\right)(-2)}\right| = \left|\frac{\frac{3}{2}}{2}\right| = \left|\frac{3}{4}\right| = 0.75$$

$\therefore\ \theta = \tan^{-1}(0.75)$

$\Rightarrow \theta = 36^\circ\ 52'$

2) Find the equation of the straight line passing through (-1, 4) and parallel to x+2y =3.

Solution:

Let the equation of line parallel to $x+2y-3 = 0$ (1)

is $x+2y+k = 0$ (2)

Equation (2) passes through (-1,4)

put x=-1, y =4 in equation (2)

$(-1) + 2(4) + k = 0$

$-1 + 8 + k = 0$

$k = -7$

$\therefore$ Required line is $x + 2y - 7 = 0$

3) Find the equation to the line through the point (3,-3) and perpendicular to $4x-3y-10 = 0$

Solution:

Required straight line is perpendicular to $4x-3y-10 = 0$ (1)

and passing through (3,-3).

$\therefore$ Required equation of the straight line is

$$-3x - 4y + k = 0 \quad (2)$$

Required line passes through (3,-3)

Put $x = 3$, $y = -3$ in equation (2)

$$-3(3) - 4(-3) + k = 0$$

$$-9+12+k = 0$$

$$3+k = 0$$

$$k = -3$$

Sub in equation (2)

$$-3x - 4y - 3 = 0$$

$\therefore$ Required equation of straight line is

$$3x + 4y + 3 = 0$$

3.2 PAIR OF STRAIGHT LINES THROUGH ORIGIN

Any line passing through the origin is of the form $ax+by = 0$

Let $a_1x + b_1y = 0$ (1)

and $a_2x + b_2y = 0$ (2)

be the two lines passing through the origin.

The combined equation of (1) and (2) is

$$(a_1x+b_1y)(a_2x+b_2y) = 0$$

$$a_1 a_2x^2 + (a_1b_2+a_2b_1)xy + b_1b_2y^2 = 0 \quad (3)$$

Taking $a_1a_2 = a$, $a_1b_2+a_2b_1 = 2h$, and $b_1b_2 = b$

We get

$$ax^2 + 2hxy + by^2 = 0 \quad (4)$$

which is a homogenous equation of second degree in x and y. It represents a pair of straight lines passing through the origin.

Let m_1 and m_2 are the slopes of the lines given by (4). Then the separate equations are

$y = m_1x$ and $y = m_2x$

(i.e.) $y - m_1x = 0$ (5)

$y - m_2x = 0$ (6)

$(y - m_1x)(y - m_2x) = 0$

$y^2 - (m_1+m_2)\,xy + m_1m_2x^2 = 0$

(i.e.) $m_1m_2x^2 - (m_1+m_2)xy + y^2 = 0$ (7)

Equation (4) and (7) represent the same part of straight lines. Hence the ratios of the corresponding co-efficient of like terms are proportional.

$$\frac{m_1 m_2}{a} = \frac{-(m_1 + m_2)}{2h} = \frac{1}{b} \qquad (8)$$

The relation (8) gives

$$m_1+m_2 = \frac{-2h}{b} \qquad (9)$$

i.e., Sum of the slopes = $\frac{-2h}{b}$

and $m_1m_2 = \frac{a}{b}$

i.e., product of the slopes = $\frac{a}{b}$

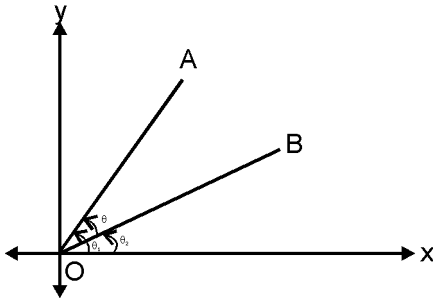
BOOK WORK :

Find the angle between the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ passing through origin. Also derive the conditions for the two separate lines to be (i) perpendicular (ii) coincident (or parallel).

Proof:

We know angle between two straight lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



$$\tan\theta = \pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

$$= \pm\frac{\sqrt{\left[\frac{-2h}{b}\right]^2 - 4\left[\frac{a}{b}\right]}}{1 + \frac{a}{b}}$$

$$= \pm\frac{\sqrt{\frac{4h^2}{b^2} - 4\left[\frac{a}{b}\right]}}{\frac{a + b}{b}}$$

$$= \pm\frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{a + b}{b}}$$

$$= \pm\frac{\sqrt{4(h^2 - ab)}}{b} \text{ x } \frac{b}{a + b}$$

$\tan\theta = \pm 2\dfrac{\sqrt{h^2 - ab}}{a+b}$

(i.e.) $\theta = \tan^{-1}\left[\pm 2\dfrac{\sqrt{h^2 - ab}}{a+b}\right]$

is the angle between the pair of straight lines.

(iii) **Condition for the two straight lines to be perpendicular**

If the two lines are perpendicular, then $\theta = 90^\circ$

$\therefore \tan\theta = \tan 90^\circ$

$\pm 2\dfrac{\sqrt{h^2 - ab}}{a+b} = \infty = \dfrac{1}{0}$

$\therefore a + b = 0$

(i.e.) coefficient of x^2 + coefficient of y^2 = 0

(iv) **Condition for the two straight lines to be coincident**

If the two straight lines are coincident

then $\theta = 0$

$\therefore \tan\theta = \tan 0$

$\pm 2\dfrac{\sqrt{h^2 - ab}}{a+b} = 0$

(i.e.) $h^2 - ab = 0$

(i.e.) $h^2 = ab$

3.2 WORKED EXAMPLES

PART - A

1) Write down the combined equation of the pair of lines x-2y =0 and 3x+2y = 0

Solution:

The combined equation is $(x-2y)(3x+2y) = 0$

(i.e.) $3x^2 + 2xy - 6xy - 4y^2 = 0$

(i.e.) $3x^2 - 4xy - 4y^2 = 0$

2) Write down the separate equations of the pair of lines $12x^2 + 7xy - 10y^2 = 0$

Solution:

$12x^2 + 7xy - 10y^2 = 0$

$12x^2 + 15xy - 8xy - 10y^2 = 0$

$12x^2 - 8xy + 15xy - 10y^2 = 0$

$4x(3x-2y) + 5y(3x-2y) = 0$

$(3x-2y)(4x+5y) = 0$

∴ The separate equations are $3x-2y = 0$ and $4x+5y = 0$

3) Show that the two lines represented by $4x^2 + 4xy + y^2 = 0$ are parallel to each other.

Solution:

$$4x^2 + 4xy + y^2 = 0 \quad (1)$$

This is of the form $ax^2 + 2hxy + by^2 = 0$

Here a =4, 2h =4, h=2, b=1

If the lines are parallel then $h^2-ab=0$

$$\begin{aligned} h^2 - ab &= 2^2 - (4)(1) \\ &= 4 - 4 = 0 \end{aligned}$$

∴ pair of lines are parallel

4) Find the value of 'p' if the pair of lines $4x^2 + pxy + 9y^2 = 0$ are parallel to each other.

Solution:

$$4x^2 + pxy + 9y^2 = 0$$

This is of the form $ax^2+2hxy+by^2=0$

Here $a = 4$, $2h = p$, $h = \frac{p}{2}$, $b = 9$

If the lines are parallel

$$h^2 - ab = 0$$

$$\left(\frac{p}{2}\right)^2 - (4)(9) = 0$$

$$\frac{p^2}{4} - 36 = 0$$

$$p^2 = 144$$

$$\therefore\ p = \pm 12$$

5) Prove that the lines represented by $7x^2-48xy-7y^2 = 0$ are perpendicular to each other.

Solution:

$$7x^2 - 48xy - 7y^2 = 0$$

This is of the form $ax^2+2hxy+by^2=0$

Here $a = 7$, $2h=-48$, $h=-24$, $b=-7$

If the lines are perpendicular $a+b=0$

(i.e.) $7-7 = 0$

6) If the two straight lines represented by the equation $px^2-5xy+7y^2 = 0$ are perpendicular to each other, find the value of p.

$px^2+48xy+7y^2 = 0$

Solution:

$$px^2+48xy+7y^2 = 0$$

This is of the form $ax^2+2hxy+by^2=0$

Here a=p, b =7

If the lines are perpendicular

a + b = 0

(i.e.) p+7 = 0

p = -7

PART – B

1) Find the separate equations of the line $2x^2 - 7xy + 3y^2 = 0$. Also find the angle between them.

Solution:

$$2x^2 - 7xy + 3y^2 = 0$$

This is of the form $ax^2 + 2hxy + by^2 = 0$

(a=2, 2h = -7, h=-7/2, b=3)

$$2x^2 - 6xy - xy + 3y^2 = 0$$

$$2x(x - 3y) - y(x - 3y) = 0$$

$$(x - 3y)(2x - y) = 0$$

$\therefore$ The separate equations are

x - 3y = 0 and 2x-y = 0

Let θ be the angle between the two straight lines

$$\therefore \tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

$$= \pm\frac{2\sqrt{(-7/2)^2 - (2)(3)}}{2 + 3}$$

$$= \pm 2\frac{\sqrt{\frac{49}{4} - 6}}{5}$$

$$=\pm 2\frac{\sqrt{\frac{49-24}{4}}}{5}$$

$$=\pm 2\frac{\sqrt{\frac{25}{4}}}{5}$$

$$=\pm\frac{2x\frac{5}{2}}{5}$$

$\tan\theta = \pm 1$

$\tan\theta = \tan 45$, $\therefore\ \theta = 45^\circ$

2) The slope of one of the lines $ax^2+2hxy+by^2=0$ is thrice that of the other. Show that $3h^2=4ab$

Solution:

$$ax^2+2hxy+by^2=0 \qquad (1)$$

Let $y=m_1x$ and $y=m_2x$ be the separate equations of equation (1)

$$m_1+m_2=\frac{-2h}{b} \qquad (2)$$

$$m_1m_2=\frac{a}{b} \qquad (3)$$

Slope of one of the line = thrice slope of the other line

(i.e.) $m_1 = 3m_2$

Equation (2) becomes

$$3m_2+m_2=-\frac{2h}{b}$$

$$4m_2=\frac{-2h}{b}$$

$$m_2=\frac{-2h}{4b}=\frac{-h}{2b}$$

Substitute $m_1=3m_2$ in equation (3)

$$3m_2.m_2 = \frac{a}{b}$$

$$(3m_2)^2 = \frac{a}{b}$$

$$3\left(\frac{-h}{2b}\right)^2 = \frac{a}{b}$$

$$3\left(\frac{-h^2}{4b^2}\right) = \frac{a}{b}$$

$$\frac{3h^2}{4b^2} = \frac{a}{b}$$

$$3h_2b = 4ab_2$$

$$(ie)3h^2 = 4ab$$

3.3 PAIR OF STRAIGHT LINES NOT PASSING THROUGH THE ORIGIN

Consider the second degree equation

$$(lx + my + n)(l'x + m'y + n') = 0 \qquad (1)$$

If (x_1, y_1) lies on $lx + my + n = 0 \qquad (2)$

then $lx_1 + my_1 + n = 0$. Hence (x_1,y_1) satisfies equation (1).

Similarly any point on $l'x + m'y + n' = 0 \qquad (3)$

also satisfies (1)

conversely, any point which satisfies (1) must be on any of the straight lines (2) and (3). Thus $(lx + my + n)(l'x + m'y + n') = 0$ represent a pair of lines.

Expanding equation (1) we get

$$ll'x^2 + lm'xy + lxn' + l'mxy + mm'y^2 + mn'y + l'xn + nm'y + nn' = 0$$

$$ll'x^2 + xy(lm' + l'm) + mm'y^2 + (ln' + l'n)x + (mn' + m'n)y + nn' = 0$$

Taking $a = ll'$ $\quad 2h = lm' + l'm$, $\quad b = mm'$

$2g = ln' + l'n$ $\quad 2f = mn' + m'n$ $\quad c = nn'$

We get $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Condition for the second degree equation ax^2+2hxy+by^2+ 2hx+2fy+c to represent a pair of straight lines is abc+2fgh-af^2-bg^2-ch^2

(or)

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

1) Angle between pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\tan\theta = \pm 2\frac{\sqrt{h^2 - ab}}{a + b}$$

2) The condition for the pair of lines to be parallel is $h^2 - ab = 0$

3) The condition for the pair of lines to be perpendicular is a + b=0.

3.3 WORKED EXAMPLES

PART - A

1) Find the combined equation of the lines whose separate equations are $2x - 3y + 2 = 0$ and $4x + y + 3 = 0$

Solution:

The two separate lines are $2x - 3y + 2 = 0$ and $4x + y + 3 = 0$

The combined equation of the given line is

$(2x - 3y + 2)\,(4x + y + 3) = 0$

$8x^2 + 2xy + 6x - 12xy - 3y^2 - 9y + 8x + 2y + 6 = 0$

(i.e.) $8x^2 - 10xy - 3y^2 + 14x - 7y + 6 = 0$

2) Show that the pair of lines given by $9x^2+24xy+16y^2+21x$ $+28y+6=0$ are parallel.

Solution:

This is of the form $ax^2+2hxy+by^2+2gx+2fy+c=0$

Here a = 9, 2h = 24, b=16

h = 12

If the lines are parallel $h^2-ab=0$

(ie) $(12)^2-(9)(16)=0$

$=144-144=0$

Hence the lines are parallel.

3) Show that the pair of lines given by $6x^2+3xy-6y^2-8x$ $+5y-3=0$ are perpendicular

Solution:

This is of the form $ax^2+2hxy+hy^2+2gx+2fy+c=0$

Here a = 6 b=-6

If the lines are perpendicular

a+b = 0

(i.e) 6 + (-6) = 0

Hence the pair of lines are perpendicular.

PART – B

1) Prove that equation $6x^2+13xy+6y^2+8x+7y+2=0$ represents a pair of straight lines.

Solution:

Given equation

$$6x^2+13xy+6y^2+8x+7y+2=0 \qquad (1)$$

(i.e.) This is of the form

$$ax^2+2hxy+by^2+2gx+2fy+l=0$$

Hence a = 6 b=6 c=2

2h=13, 2g = 8, 2f = 7

If the equation (1) represents a pair of straight lies then

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\text{LHS} = \begin{vmatrix} 12 & 13 & 8 \\ 13 & 12 & 7 \\ 8 & 7 & 4 \end{vmatrix}$$

$$= 12(48-49) - 13(52-56) + 8(91-96)$$

$$= -12 + 52 - 40$$

$$= 0 = \text{RHS}$$

Hence the given equation represents a pair of straight lines.

2) Show that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines. Also find the separate equation of the lines.

Solution:

Given $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$

This is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

a=3 b=2 c=2

2h=7 2g=5 2f=5

If the equation (1) represents a pair of straight lies then

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\text{LHS} = \begin{vmatrix} 6 & 7 & 5 \\ 7 & 4 & 5 \\ 5 & 5 & 4 \end{vmatrix}$$

$$= 6(16-25) - 7(28-25) + 5(35-20)$$

$= -54 - 21 + 75$

$= 0 = \text{RHS}$

$\therefore$ The given equation represents a pair of straight lines.
Next we find separate lines.

Factorise the second degree terms

Let $3x^2 + 7xy + 2y^2 = 3x^2 + 6xy + xy + 2y^2$

$= 3x(x + 2y) + y(x + 2y)$

$= (x + 2y)(3x + y)$

$\therefore\ 3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = (3x + y + l)(x + 2y + m)$ (say)

Equating the coefficient of x, l+3m=5 (2)

Equating the coefficient of y, 2l+m=5 (3)

Solving (2) and (3)

2l+6m = 10

2l+m = 5

5m=5

m=1

Sub in (2),

l + 3(1) = 5

l = 2

$\therefore$ The separate equation are 3x+y+2 =0 and x+2y+1=0

3) Find 'k' if $2x^2-7xy+3y^2+5x-5y+k = 0$ represents a pair of straight lines. Find the angle between them.

Solution :

$$2x^2 - 7xy + 3y^2 + 5x - 5y + k = 0$$

This is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Hence a=2 b=3 c=k

2h=-7 2g=5 2f=-5

Since the given equation represents a pair of straight lines

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 4 & -7 & 5 \\ -7 & 6 & -5 \\ 5 & -5 & 2k \end{vmatrix} = 0$$

$$4(12k - 25) + 7(-14k + 25) + 5(35 - 30) = 0$$

48k – 100 – 98k + 175 + 25 = 0

-50k + 100 = 0

-50k = - 100

K = 2

It 'θ' is the angle between the given lines then

$$\tan\theta = \pm 2\frac{\sqrt{h^2 - ab}}{a + b}$$

$$= \pm 2\frac{\sqrt{\left(\frac{-7}{2}\right)^2 - (2)(3)}}{2 + 3}$$

$$= \pm 2\frac{\sqrt{\frac{49}{4} - 6}}{5}$$

$$= \pm 2\frac{\sqrt{\frac{25}{4}}}{5}$$

$$= \pm 2\frac{\left(\frac{5}{2}\right)}{5}$$

tan θ = 1

tan θ = tan 45

$$\therefore \theta = \frac{\pi}{4}$$

4) Show that the pair straight lines $4x^2+4xy+y^2-6x-3y-4=0$ represents a pair of parallel straight lines and find the distance between them.

Solution:

Given: $4x^2+4xy+y^2-6x-3y-4=0$ (1)

This is of the form $ax^2+2hxy+by^2+2gx+2fy+c=0$

Here a = 4	2h = 4	h=2
b=1	2g=-6	g=-3
2f=-3	f=-3/2	c=-4

If the lines are parallel $h^2-ab=0$

(ie) $(2)^2$ (4) (1) = 0

4-4 = 0

$\therefore$ The given equation (1) represents a pair of parallel straight lines.

To find the separate lines of (1)

Factorise $4x^2+4xy+4^2$

$$4x^2+2xy+2xy+y^2 = 2x(2x+y)+y(2x+y)$$
$$=(2x+y)(2x+y)$$
$$=(2x+y)^2$$

$4x^2+4xy+y^2-6x-3y-4=0$

$(2x+y)^2-3(2x+y)-4=0$

Let z = $2x+y$, then

(i.e.) $z^2-3z-4=0$

$(z+1)(z-4)=0$

$z+1=0$ and $z-4=0$

(i.e.) $2x+y+1=0$ and $2x+y-4=0$ are the separate equations

Distance between parallel lines

$2x+y-4=0$ and $2x+y+1=0$ is

$$\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

$$\frac{|-4 \;\; -1|}{\sqrt{(2)^2 + (1)^2}} \qquad = \frac{5}{\sqrt{5}} \quad = \sqrt{5}$$

EXERCISE

PART – A

1) Find the perpendicular distance from the point (3,-3) to the line 2x+4y+2=0
2) Find the perpendicular distance from the point (2,6) to the line x-4y+6=0
3) Find the length of the perpendicular to the line x-6y+5 = 0 from the origin.
4) Find the distance between the line 3x+3y+4=0 and 3x+3y-2 =0
5) Find the distance between the line x+2y-19=0 and x+2y-31 =0
6) Show that the lines 3x+2y-5=0 and 6x+4y-8 = 0 are parallel.
7) Show that the lines 2x-6y+6 = 0 and 4x-12y+7 = 0 are parallel.
8) Find the value of 'k' if the lines 7x-2y+13 = 0 and kx=3y+8 are parallel.
9) Find the value of 'p' if the lines 5x+3y = 6 and 3x+py = 7 are parallel.
10) Show that the lines 4x-3y=0 and 3x+4y+8 = 0 are perpendicular.
11) Show that the lines 4x-2y+6 = 0 and 2x+4y-4=0 are perpendicular
12) Find the value of 'p' if the lines 3x-py-4=0 and 2x+3y=7 are perpendicular.
13) Find the value of 'p' if the lines 2x-py+6=0 and 3x-2y+8 = 0 are perpendicular.

14) Find the slope of the line parallel to the line joining the points (3,4) and (-4,6)

15) Find the slope of the line perpendicular to the line joining the points (3,1) and (-4,3)

16) Show that the line joining the points (3,-5) and (-5,-4) is parallel to the line joining (7,10) and (15,9)

17) Show that the line joining the points (2,-2) and (3,0) is perpendicular to the line joining (2,2) and (4,1).

18) Find the equation of the line passing through (2,4) and parallel to the line $x+3y+7=0$

19) Find the equation of the line passing through (-2,5) and perpendicular to $5x-3y+8 = 0$

20) Write down the combined equation of the lines whose separate equation are

(i) $4x + 2y = 0$ and $2x-y =0$

(ii) $3x + 2y = 0$ and $2x – y = 0$

(iii) $x + 2y = 0$ and $3x + 2y = 0$

(iv) $x + 2y = 0$ and $2x – y = 0$

21) Find the separate equation of each of the straight liens represents by

(i) $9x^2 – 16y^2 = 0$

(ii) $2x^2 – 5xy+2y^2 = 0$

(iii) $6x^2+xy-y^2=0$

(iv) $15x^2+17xy+2y^2=0$

22) Show that the two lines represented by $9x^2 + 6xy+y^2 = 0$ are parallel to each other.

23) Show that the equation $4x^2 – 12xy+9y^2 = 0$ represents a pair of parallel straight lines.

24) Find the values of p if the two straight lines represented by $20x^2+pxy +5y^2 = 0$ are parallel to each other.

25) Show that the pair of straight lines given by $2x^2-3xy-2y^2=0$ is perpendicular.

26) Find the value of 'p' so that the two straight lines represented by $px^2+6xy-y^2 = 0$ are perpendicular to each other.

27) Write down the combined equation of the lines whose separate equations are

(i) $x+2y = 0$ and $2x-y+1=0$

(ii) $x+2y=10$ and $2x-y-3=0$

(iii) $x+2y-1=0$ and $3x+2y+3=0$

28) Show that the equation $4x^2+4xy+y^2-6x-3y-4 =0$ represents two parallel lines.

29) Show that the equation $x^2+6xy+9y^2+4x+12y-5 = 0$ represents two parallel straight lines

30) Show that the equation $2x^2-3xy-2y^2+2x+y = 0$ represents a perpendicular pair of straight lines

PART – B

1) Show that the following equation represents a pair of straight line

(i) $9x^2-6xy+y^2+18x-6y+8 = 0$

(ii) $9x^2 + 24xy+16y^2+21x+28y+6 = 0$

(iii) $4x^2 + 4xy+y^2 – 6x-3y-4 = 0$

2) Find the angle between the pair of straight lines

3) Find the angle between the pair of lines given by $6x^2-13xy+5y^2= 0$. Find also the separate equations.

4) Find the angle between the pair of lines given by $9x^2+12xy+4y^2=0$. Find also the separate equations.

5) Find the angle between the pair of lines given by $3x^2-8xy+5y^2=0$. Find also the separate equation.

6) Find the separate equations of the pair of lines $3x^2-4xy+y^2=0$. Find also the angle between the lines.

7) If the slope of one of the lines of $ax^2+2hxy+by^2=0$ is twice the slope of the other, show that $8h^2= 9ab$

8) If the equation $ax^2+3xy-2y^2-5x+5y+c=0$ represents two lines perpendicular to each other, find the value of 'a' and 'c'.

9) Show that the equation $12x^2-10xy+2y^2+14x-5y+2=0$ represents a pair of lines. Also find the separate equations.

10) Show that the equation $12x^2+7xy-10y^2+13x+45y-35=0$ represents a pair of straight lines. Also find the separate equations.

11) Write down the separate equations of $3x^2 – 7xy – 6y^2 – 5x +26y – 8 = 0$. Also find the angle between the lines.

12) Show that the equation $9x^2+24xy+16y^2+21x+28y+6=0$ represents a pair of parallel straight lines. Find the separate equations and the distance between them.

13) Show that the equation $9x^2-6xy+y^2+18x-6y+8=0$ represents two parallel straight lines. Find the distance between them.

ANSWERS

PART – A

1. $\frac{-4}{\sqrt{20}}$	2. $\frac{-16}{\sqrt{17}}$	3. $\frac{5}{\sqrt{37}}$	4. $\frac{2}{\sqrt{18}}$
5. $\frac{12}{\sqrt{5}}$	8. $k=\frac{21}{2}$	9. $p=\frac{9}{5}$	5. P = 2
13. P = -3	14. $\frac{-2}{7}$	15. $\frac{7}{2}$	18. X+3y-14=0

19. 3x+5y-19 = 0

20. (i) $8x^2 - 2y^2 = 0$

(ii) $6x^2 + xy - 2y^2 = 0$

(iii) $3x^2 + 8xy + 4y^2 = 0$

(iv) $2x^2 + 3xy - 2y^2 = 0$

21. (i) $(3x+4y)=0$, $3x-4y=0$ (ii) $x-2y=0, 2x-y=0$

(iii) $2x+y=0, 3x-y=0$ (iv) $x+y=0, 15x+2y=0$

24. $P = \pm 20$ 26. $P = 1$

27. (i) $2x^2+3xy-2y^2+x+2y=0$

(ii) $2x^2+3xy-2y^2-23x+4y+30=0$ (iii) $3x^2+8xy+4y^2+4y-3=0$

PART – B

(2) (i) $\theta = 60°$, (ii) $\theta = 90°$

(3) $\tan\theta = \frac{7}{11}$, $\theta = 32°, 28'$, $3x-5y=0$, $2x-y=0$

(4) $\theta=0$, $3x+2y=0$, $3x+2y=0$

(5) $\tan\theta = \frac{1}{4}$, $3x-5y=0$, $x-y=0$

(6) $3x-y=0$, $x-y=0$, $\tan\theta = \frac{1}{2}$

(8) $a=2$, $c=-3$

(9) $2x-y+2=0$, $6x-2y+1=0$

(10) $4x+5y-5=0$

$3x-2y+7 = 0$

(11) $3x+2y-8=0$

$x-3y+1=0$

$\tan\theta = \frac{11}{3}$

(12) $3x+4y+1=0$

$3x+4y+6=0$

dist = 1

(13) $\frac{\sqrt{10}}{5}$

UNIT IV

TRIGONOMETRY

4.1: Trigonometrical ratio of allied angles-Expansion of Sin(A±B) and cos(A±B) problems using above expansion

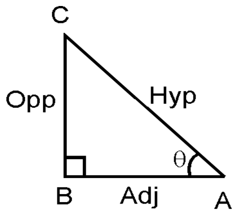
4.2: Expansion of tan(A±B) and Problems using this expansion

4.3: Trigonometrical ratios of multiple angles (2A only) and sub-multiple angles. Simple problems.

Introduction

Trigonometry is one of the oldest branches of mathematics. The word trigonometry is derived from Greek words 'Trigonon' and 'metron' means measurement of angles. In olden days Trigonometry was mainly used as a tool for studying astronomy. In earlier stages Trigonometry was mainly concerned with angles of a triangle. But now it has its applications in various branches of science such as surveying, engineering, navigations etc.,For the study of higher mathematics, knowledge of trigonometry is essential.

Trigonometrical Ratios:



In a right angled triangle ABC,

i) Sine of an angle θ = sin θ = $\frac{\text{Opposite side}}{\text{Hypotenuse}}$

ii) Cosine of an angle θ = cos θ = $\frac{\text{Adjacent side}}{\text{Hypotenuse}}$

iii) Tangent of an angle θ = tan θ = $\frac{\text{Opposite side}}{\text{Adjacent side}}$

iv) Cotangent of an angle θ = cot θ = $\frac{\text{Adjacent side}}{\text{Opposite side}}$

v) Secant of an angle θ = sec θ = $\frac{\text{Hypotenuse}}{\text{Adjacent side}}$

vi) Cosecant of an angle θ = cosec θ = $\frac{\text{Hypotenuse}}{\text{Opposite side}}$

Note:

1. Cosec θ = $\frac{1}{\sin\theta}$
2. Sec θ = $\frac{1}{\cos\theta}$
3. Cot θ = $\frac{1}{\tan\theta}$
4. Tan θ = $\frac{\sin\theta}{\cos\theta}$
5. Cot θ = $\frac{\cos\theta}{\sin\theta}$

Fundamental trigonometrical identities

1) $\text{Sin}^2\theta + \cos^2\theta = 1$
2) $1 + \tan^2\theta = \sec^2\theta$
3) $1 + \cot^2\theta = \text{cosec}^2\theta$

Trigonometricall reties of known Angels

θ	0°	30°	45^0	60^0	90^0	180^0	270^0	360^0
Sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
Tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	-∞	0

Important results:

For all values of θ

i) $\sin(-\theta) = -\sin\theta$

ii) $\cos(-\theta) = \cos\theta$

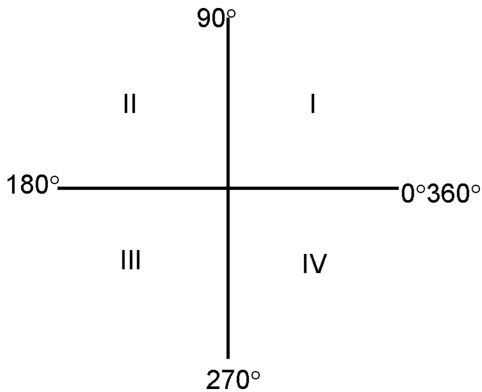
iii) $\tan(-\theta) = -\tan\theta$

iv) $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

v) $\sec(-\theta) = \sec\theta$

vi) $\cot(-\theta) = -\cot\theta$

Signs of trigonometrical ratios:



i) In the first quadrant, all trigonometrical ratios are positive.

ii) In the second quadrant, sin θ and its reciprocal cosec θ are positive. Other trigonometrical ratios are negative.

iii) In the third quadrant, tan θ and its reciprocal cot θ are positive. Other trigonometrical ratios are negative.

iv) In the fourth quadrant, cos θ and its reciprocal sec θ are positive. Other trigonometrical ratios are negative.

v) The signs of trigonometrical ratios are usually remembered by code word **"All Silver Tea Cups"** where the four words beginning with **A,S,T,C** correspond to All ratios being positive in the I quadrant, Sine in the II quadrant, Tangent in the III quadrant and Cosine in the IV quadrant respectively.

This is tabulated as follows:

Functions / Quadrants	I	II	III	IV
Sin	+	+	–	–
Cosine	+	–	–	+
Tangent	+	–	+	–
Cotangent	+	–	+	–
Secant	+	–	–	+
cosecant	+	+	–	–

4.1 TRIGONOMETRICAL RATIOS OF RELATED OR ALLIED ANGLES:

The basic angle is ‘θ’ and angles associated with ‘θ’ by a right angle (or) its multiples are called related angles or allied angles. Thus $90^0 \pm \theta$, $180^0 \pm \theta$, $270^0 \pm \theta$, $360^0 \pm \theta$ are known as related or allied angles.

Working rule:

To determine the trigonometrical ratios of any angle, follow the procedure given below:

i) Determine the sign (+ve or -ve) of the given T- ratio in the particular quadrant by observing the quadrant rule “ASTC”.

ii) Determine the magnitude of the angle by writing in the form (p x $90 \pm \theta$). If p is even (like 2,4,6,…) T – ratio of allied angle becomes the same ratio of θ. i.e. for $180 \pm \theta$ etc., no change in ratio If p is an odd (like 1,3,5,…) T.Ratio of the allied angles becomes the co-ratio of θ. i.e. for $90 \pm \theta$, $270 \pm \theta$ etc,

$\sin \leftrightarrow \cos, \tan \leftrightarrow \cot, \sec \leftrightarrow \operatorname{cosec}$

The trigonometrical ratios of related or allied angles are tabulated as follows.

Angle / Ratio	90-θ	90+θ	180-θ	180°+θ	270-θ	270+θ	360°-θ	360+θ
sin	cos θ	cos θ	sin θ	-sin θ	-cos θ	-cos θ	-sin θ	sin θ
cos	sin θ	-sin θ	-cos θ	-cos θ	-sin θ	sin θ	cos θ	cos θ
tan	cot θ	-cot θ	-tan θ	tan θ	cot θ	-cot θ	-tan θ	tan θ
cot	tan θ	-tan θ	-cot θ	cot θ	tan θ	-tan θ	-cot θ	cot θ
sec	cosec θ	-cosec	-secθ	-sec θ	-cosec θ	cosec θ	sec θ	sec θ
cosec	sec θ	sec θ	cosec θ	-cosec θ	-sec θ	-sec θ	-cosec θ	cosec θ

Note :

The trigonometrical ratios of angle n x $360^0 \pm \theta$ are same as those of $\pm \theta$

(e.g.)

i) sin 840^0

$= \sin(2 \times 360^0 + 120^0) = \sin 120^0$

$= \sin(180^0 - 60^0) = \sin 60^0 = \frac{\sqrt{3}}{2}$

ii) $\cos 600^0 = \cos(2 \times 360^0 - 120^0) = \cos(-120^0)$

$= \cos 120^0$

$= \cos(180^0 - 60^0) = \frac{1}{2}$

iii) Tan $2460^0 = \tan(6 \times 360^0 + 300^0) = \tan 300^0$

$= \tan(360^0 - 60^0) = -\tan 60^0$

$= -\sqrt{3}$

WORKED EXAMPLES

PART – A

Find the value of the following without using tables:

i) Sin 480° ii) cos (-300°)

iii) tan 765° iv) sec (-420°)

Solution:

i) $\sin 480^\circ = \sin(360^\circ + 120^\circ)$

$= \sin 120^\circ$

$= \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

ii) $\cos(-300^\circ) = \cos 300^\circ$

$= \cos(360^\circ - 60^\circ)$

$= \cos 60^\circ = \frac{1}{2}$

iii) $\tan 765^\circ = \tan(2 \times 360^\circ + 45^\circ)$

$= \tan 45^\circ = 1$

iv) $\sec(-420^\circ) = \sec 420^\circ$

$= \sec(360^\circ + 60^\circ) = \sec 60^\circ = 2$

2. Prove that $\sin(-330^0) \sin 420^0 = \frac{\sqrt{3}}{4}$

Solution

LHS = $-\sin 330^\circ \sin 420^\circ$

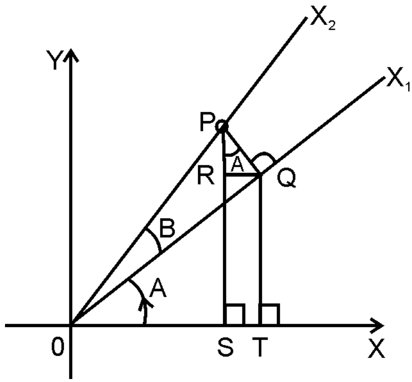
$= -\sin(360^\circ - 30^\circ) \sin(360^\circ + 60^\circ)$

$= \sin 30^\circ \sin 60 = \frac{1}{2}, \sqrt{3}/2 = \sqrt{3}/4$

4.2 COMPOUND ANGLES

If an angle is expressed as the algebraic sum or difference of two or more angles, then it is called compound angle.

E.g: A + B, A – B, A + B + C, A + B – C etc…are compound angles.



Prove Geometrically that

1. sin(A+B)=sinAcosB+cosA sinB

2. cos(A+B)=cosA cosB-sinA sinB

proof:

Draw $OX_1$and OX_2 such that $\lfloor X_1OX = A$ and $\lfloor X_2OX_1 = B$

P is any point on OX_2. Draw perpendiculars PQ to OX, and PS to OX from the point P. Draw perpendiculars QR to PS and QT to OX from the point Q.

(i) In the right angled triangle OPS

$$\sin(A+B) = \frac{\text{opposite side}}{\text{Hypotenuse}}$$

$$= \frac{PS}{OP}$$

$$= \frac{PR+RS}{OP} = \frac{PR+QT}{OP}$$

$$= \frac{PR}{OP} + \frac{QT}{OP}$$

$$= \frac{PR}{PQ}\frac{PQ}{OP} + \frac{QT}{OQ}\frac{OQ}{OP}$$

= cosA sinB+ sinA cosB

$\therefore$ sin (A+B) = sinA cosB+cosA sinB

(ii) In the right angled triangle OPS

$$\cos(A+B) = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$= \frac{OS}{OP}$$

$$= \frac{OT - ST}{OP} = \frac{OT - RQ}{OP}$$

$$= \frac{OT}{OP} - \frac{RQ}{OP}$$

$$= \frac{OT}{OQ}\frac{OQ}{OP} - \frac{RQ}{PQ}\frac{PQ}{OP}$$

=cosA cosB-sinA sinB

$\therefore$ cos(A+B) =cosA CosB-sinA sinB

Formula:

1) Sin (A + B) = Sin A Cos B + Cos A Sin B
2) Sin (A – B) = Sin A Cos B – Cos A Sin B
3) Cos (A + B) = Cos A Cos B – Sin A Sin B
4) Cos (A – B) = Cos A Cos B + Sin A Sin B

Result:

(i) Prove that sin (A + B) sin (A – B) = $\sin^2 A - \sin^2 B$

Solution

LHS =sin (A +B) sin (A – B)

=(sin A cos B + cos A sin B) (sin A cos B – cos A sin B)

=$(\sin A \cos B)^2 - (\cos A \sin B)^2$

=$\sin^2 A \cos^2 B - \cos^2 A \sin^2 B)$

=$\sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$

=$\sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$

=$\sin^2 A - \sin^2 B$

$\therefore$Sin (A + B) sin (A – B) = $\sin^2 A - \sin^2 B$

(ii) Prove that cos (A + B) cos (A – B) = $\cos^2 A - \sin^2 B$

Solution

LHS $=\cos(A+B)\cos(A-B)$

$=(\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$=(\cos A\cos B)^2 - (\sin A\sin B)^2$

$=\text{Cos}^2 A\cos^2 B - \sin^2 A\sin^2 B$

$=\text{Cos}^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$

$=\text{Cos}^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B$

$=\text{Cos}^2 A - \text{Sin}^2 B$

$\therefore \text{Cos}(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

Also $\text{Cos}(A+B)\cos(A-B) = \cos^2 B - \sin^2 A$

4.1 WORKED EXAMPLES

PART – A

1.) Find the value of $\sin 65^0 \cos 25^0 + \cos 65^0 \sin 25^0$

Solution:

$\text{Sin}65^0 \cos 25^0 + \cos 65^0 \sin 25^0 = \sin(65^0 + 25^0)$

$= \sin 90^0 \quad = 1$

2.) Find the values of $\sin 40^0 \cos 10^0 - \cos 40^0 \sin 10^0$

Solution:

$\text{Sin}40^0 \cos 10^0 - \cos 40^0 \sin 10^0 = \sin(40^0 - 10^0)$

$= \sin 30^0 \quad = \frac{1}{2}$

3.) What is the value of $\cos 50^0 \cos 40^0 - \sin 50^0 \sin 40^0$

Solution:

$\text{Cos}50^0 \cos 40^0 - \sin 50^0 \sin 40^0 = \cos(50^0 + 40^0)$

$= \cos 90^0 \quad = 0$

4.) What is the value of $\cos 70^0 \sin 10^0 + \sin 70^0 \sin 10^0$

Solution:

$\text{Cos}70^0 \cos10^0 + \sin70^0 \sin10^0 = \cos(70^0 - 10^0)$

$= \cos 60^0 \qquad = \frac{1}{2}$

5.) Find the value of $\sin15^0$

Solution:

$\text{Sin}15^0 = \sin(45^0 - 30^0)$

$= \sin45^0 \cos30^0 - \cos45^0 \sin30^0$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \qquad = \frac{\sqrt{3}-1}{2\sqrt{2}}$

6.) Find the value of $\cos75^0$

Solution:

$\text{Cos}75^0 = \cos(45^0 + 30^0)$

$= \cos45^0 \cos30^0 - \sin45^0 \sin30^0$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

7.) Prove that $\cos(60^0 - A) \cos(30^0 + A) - \sin(60^0 - A) \sin(30^0 + A) = 0$

Solution:

We have $\cos A \cos B - \sin A \sin B = \cos(A + B)$

$\text{LHS} = \cos(60^0 - A) \cos(30^0 + A) - \sin(60^0 - A) \sin(30^0 + A)$

$= \cos[(60 - A) + (30 + A)]$

$= \cos(60 - A + 30 + A) = \cos 90^0 = 0 = \text{RHS}$

8.) Prove that $\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \tan A$

Solution:

$$\text{LHS} = \frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}$$

$$= \frac{\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B}{\cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B}$$

$$= \frac{2\sin A\cos B}{2\cos A\cos B} = \frac{\sin A}{\cos A} = \tan A = \text{R.H.S}$$

9.) Prove that sin(A+B) sin(A-B)+sin(B+C)sin(B-C) +sin(C+A) sin(C-A)=0

Solution:

LHS = sin(A+B) sin(A-B)+sin(B+C)sin(B-C) +sin(C+A) sin(C-A)

= $\sin^2 A - \sin^2 B + \sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A$

= 0 RHS

PART –B

1) If A and B are acute and if $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$ prove that $A + B = \frac{\pi}{4}$

Solution:

Given : $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$

$$\cos A = \sqrt{1-\sin^2 A}$$

$$= \sqrt{1-\frac{1}{10}}$$

$$= \sqrt{\frac{9}{10}}$$

$$\cos A = \frac{3}{\sqrt{10}}$$

$$\cos B = \sqrt{1 - \sin^2 B}$$

$$= \sqrt{1 - \frac{1}{5}}$$

$$= \sqrt{\frac{4}{5}}$$

$$\cos B = \frac{2}{\sqrt{5}}$$

Sin (A+B) = sin A Cos B + Cos A sin B

$$= \frac{1}{\sqrt{10}} . \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} . \frac{1}{\sqrt{5}}$$

$$= \frac{2}{\sqrt{50}} + \frac{3}{\sqrt{50}}$$

$$= \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}}$$

$$\text{Sin}\,(A + B) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin(A + B) = \sin\ 45^\circ$$

$$\Rightarrow A + B = 45^\circ$$

$$\Rightarrow A + B = \frac{\pi}{4}$$

2) If A and B are acute and if cos A= $\frac{7}{4}$ and cosB= $\frac{13}{14}$,

Prove that $A - B = 60^\circ$ or $\frac{\pi}{3}$

Sin A $= \sqrt{1 - \cos^2 A}$

$$= \sqrt{1 - \frac{1}{49}}$$

$$= \sqrt{\frac{48}{49}}$$

$$= \frac{4\sqrt{3}}{7}$$

$$\text{Sin B} \quad = \sqrt{1 - \cos^2 B}$$

$$= \sqrt{1 - \frac{169}{196}}$$

$$= \sqrt{\frac{27}{196}}$$

$$= \frac{3\sqrt{3}}{14}$$

cos (A-B) = Cos A Cos B + sin A sin B

$$= \frac{1}{7}\ \frac{13}{14} + \frac{4\sqrt{3}}{7}\ \frac{3\sqrt{3}}{14}$$

$$= \frac{13}{98} + \frac{36}{98}$$

$$= \frac{49}{98} = \frac{1}{2}$$

Cos (A – B) = cos 60°

$\Rightarrow$ A – B = 60°

4.2 COMPOUND ANGLES (CONTINUED)

Formula:

1) tan (A+B) $= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

2) tan (A-B) $= \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

4.2 WORKED EXAMPLES

PART –A

1) Find the value of $\dfrac{\tan 20^\circ + \tan 25^\circ}{1-\tan 20^\circ \tan 25^\circ}$

Solution:

$$\frac{\tan 20^\circ + \tan 25^\circ}{1-\tan 20^\circ \tan 25^\circ} = \tan 45^\circ = 1$$

2) If tan A = $\dfrac{1}{2}$ and tan B = $\dfrac{1}{3}$, find the value of tan (A+B)

Solution :

Given tan A = $\dfrac{1}{2}$ and tan B = $\dfrac{1}{3}$,

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\frac{1}{3}} = \frac{\frac{3+2}{6}}{\frac{6-1}{6}}$$

$$= \frac{5/6}{5/6} = 1$$

3) With out using tables, find the value of tan 105°

Solution:

tan 105° = tan (60°+45°)

$$= \frac{\tan 60^\circ + \tan 45^\circ}{1-\tan 60^\circ \tan 45^\circ}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

PART –B

1) If A+B = 45°, prove (1+tan A) (1+tan B) = 2 and hence deduce the value of tan $22\frac{1}{2}^{\circ}$

Solution:

Given : A+B= 45°

Taking 'tan' on both sides

Tan (A+B)= tan 45°

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

tan A+ tan B =1-tan A tan B

tan A+ tan B+ tan A tan B = 1 (1)

RHS = (1+tan A) (1+tan B) = 1+tan A +tan B +tan A tan B

= 1+1

= 2

Thus (1+tan A) (1+tan B) = 2 (2)

Deduction :

Put B = A in A+B=45° A+A = 45° 2A = 45° A = $22\frac{1}{2}^{\circ}$

B = $22\frac{1}{2}^{\circ}$ ($\therefore$A=B)

From (2)

$(1+\tan 22\frac{1}{2}^{\circ})(1+\tan 22\frac{1}{2}^{\circ}) = 2$

$(1+\tan 22\frac{1}{2}^{\circ})^2 = 2$

$1+\tan 22\frac{1}{2}^{\circ} = \sqrt{2}$

$\tan 22\frac{1}{2}^{\circ} = \sqrt{2} - 1$

2) If A+B+C = 180°,Prove that $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+$

$\tan\frac{C}{2}\tan\frac{A}{2}=1$

Solution :

Given: A+B+C =180°

$$\Rightarrow \frac{A}{2}+\frac{B}{2}+\frac{C}{2} = 90^\circ$$

$$\Rightarrow \frac{A}{2}+\frac{B}{2} = 90^\circ - \frac{C}{2}$$

Taking tan on both sides

$$\tan\left(\frac{A}{2}+\frac{B}{2}\right) = \tan\left(90^\circ-\frac{C}{2}\right)$$

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}} = \cot\frac{C}{2}$$

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$$

$$\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) = 1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2} = 1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

3) If A+B+C =180°, prove that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

Solution:

In the above example (2), we know that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

divided by both sides by $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

$$\frac{1}{\tan\frac{C}{2}}+\frac{1}{\tan\frac{A}{2}}+\frac{1}{\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}\tan\frac{A}{2}\tan\frac{B}{2}}$$

$$\cot\frac{A}{2}+\cot+\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

4) If A&B are acute angles and if $\tan A=\frac{n}{n+1}$ and $\tan B=\frac{1}{2n+1}$, show that $A+B=\frac{\pi}{4}$

Solution:

Given: $\tan A=\frac{n}{n+1}$, $\tan B=\frac{1}{2n+1}$,

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$=\frac{\frac{n}{n+1}+\frac{1}{2n+1}}{1-\left(\frac{n}{n+1}\right)\left(\frac{1}{2n+1}\right)}$$

$$=\frac{\frac{n(2n+1)+1(n+1)}{(n+1)(2n+1)}}{\frac{(n+1)(2n+1)-n}{(n+1)(2n+1)}}$$

$$=\frac{2n^2+n+n+1}{2n^2+n+2n+1-n}$$

$$=\frac{2n^2+2n+1}{2n^2+2n+1}$$

tan (A+B) = 1

tan (A+B) = tan 45°

$\Rightarrow$ A+B = 45°

A+B = $\frac{\pi}{4}$

5) If tan A – tan B = p and cot B – cot A=q show that

$$\cot(A-B) = \frac{1}{p} + \frac{1}{q}$$

Solution:

$$\text{Given: } \left.\begin{aligned} \tan A - \tan B &= p \\ \cot B - \cot A &= q \end{aligned}\right\} \qquad (1)$$

R.H.S = $\frac{1}{p} + \frac{1}{q}$

$$= \frac{1}{\tan A - \tan B} + \frac{1}{\cot B - \cot A} \text{ using } \qquad (1)$$

$$= \frac{1}{\tan A - \tan B} + \frac{1}{\frac{1}{\tan B} - \frac{1}{\tan A}}$$

$$= \frac{1}{\tan A - \tan B} + \frac{\tan A \tan B}{\tan A - \tan B}$$

$$= \frac{1 + \tan A \tan B}{\tan A - \tan B}$$

$$= \frac{1}{\tan(A - B)}$$

= cot (A-B)

= L.H.S

4.3 MULTIPLE ANGLES OF 2A ONLY AND SUB – MULTIPLE ANGLES

Trigonometric ratios of multiple angles of 2A in terms to that of A.

1) Consider sin (A+B) = sin A cos B + cos A sin B

   Put B = A, we get

   Sin (A+A) = sinA cosA + cos A sin A

   sin 2A = 2sin A cos A

2) Consider cos (A+B) = cos A cos B – sin A sin B

   Put B = A ,we get

   cos (A+A) = cos A cos A – sin A sin A

   $\cos 2A = \cos^2 A - \sin^2 A$

   We have $\cos 2A = \cos^2 A - \sin^2 A$

   $= 1 - \sin^2 A - \sin^2 A$

   $\cos 2A = 1 - 2\sin^2 A$

   $\cos 2A = \cos^2 A - \sin^2 A$

   $= \cos^2 A - (1 - \cos^2 A)$

   $\cos 2A = 2\cos^2 A - 1$

Note: (a) $\sin^2 A = \dfrac{1-\cos 2A}{2}$

(b) $\cos^2 A = \dfrac{1+\cos 2A}{2}$

(c) $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

Consider , $\tan (A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Put B = A, we get

$$\tan (A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

To express sin 2A and cos 2A in terms of tan A:

we have sin 2A = 2sin A cos A

$$= \frac{2\sin A}{\text{Cos}A}\cos^2 A \text{ (multiple\& dividedly cosA)}$$

$$= 2\tan A \times \frac{1}{\sec^2 A}$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

Also, cos 2A $= \cos^2 A - \sin^2 A$

$$= \cos^2 A\left(1-\frac{\sin^2 A}{\cos^2 A}\right)$$

$$= \cos^2 A\ (1-\tan^2 A)$$

$$= \frac{1}{\sec^2 A}\ (1-\tan^2 A)$$

$$\text{Cos}2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

SUB-MULTIPLE ANGLES:

If A is any angle, then A/2 is called sub multiple angle.

i) Sin A = sin (2xA/2)

Sin A = 2sin A/2 cos A/2

ii) Cos A = cos [2xA/2]

Cos A = $\cos^2$ A/2 – sin A/2

= $1-2\sin^2 A/2$

(or)

= $2\cos^2 A/2-1$

Note :

$$\sin^2 A/2 = \frac{1-\cos A}{2}$$

$$\cos^2 A/2 = \frac{1+\cos A}{2}$$

$$\tan^2 A/2 = \frac{1-\cos A}{1+\cos A}$$

iii) Tan A = tan (2.A/2)

$$\text{Tan A} = \frac{2\tan A/2}{1-\tan^2 A/2}$$

Similarly, $\sin A = \dfrac{2\tan A/2}{1+\tan^2 A/2}$

$$\text{Cos A} = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}$$

4.3 WORKED EXAMPLES

PART –A

1) Find the value of $\dfrac{2\tan 15^\circ}{1+\tan^2 15^\circ}$

Solution:

We have, $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$ (1)

Put A = 15° in (1)

$$\frac{2\tan 15^\circ}{1+\tan^2 15^\circ} = \sin 2(15^\circ)$$

$$= \sin 30^\circ = \frac{1}{2}$$

2) Find the value of $\dfrac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}$

Solution:

We have $\text{Cos } 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$ (1)

Put A = 15° in (1)

$$\frac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ} = \cos 2(15^\circ)$$

$$= \cos 30^\circ \quad = \frac{\sqrt{3}}{2}$$

3) Prove that $\dfrac{\sin 2A}{1+\cos 2A}$ = tan A

Solution:

$$\text{L.H.S} = \frac{\sin 2A}{1+\cos 2A}$$

$$= \frac{2\sin A\cos A}{2\cos^2 A} \quad \because \left[\cos^2 A = \frac{1+\cos 2A}{2}\right]$$

= tanA

= R.H.S

4) Prove that $\cos^4 A - \sin^4 A = \cos 2A$

Solution:

LHS = $\cos^4 A - \sin^4 A$

= $(\cos^2 A)^2 - (\sin^2 A)^2$

= $(\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$

= (1) cos 2A

= cos 2A

5) Prove that $\dfrac{\sin A}{1-\cos A}$ = cot A/2

Solution:

$$\text{LHS} = \frac{\sin A}{1-\cos A} = \frac{2\sin A/2\cos A/2}{2\sin^2 A/2}$$

$$= \frac{\cos A/2}{\sin A/2}$$

= cot A/2 = RHS

6) Prove that $(\sin A/2 - \cos A/2)^2 = 1-\sin A$

Solution:

LHS = $(\sin A/2 - \cos A/2)^2$

= $\sin^2 A/2 + \cos^2 A/2 - 2\sin A/2 \cos A/2$

= 1-sin A

= RHS

PART –B

1) Prove that $\frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A}$ =tan A

Solution:

LHS = $\frac{\sin A + 2\sin A \cos A}{1 + \cos A + 2\cos^2 A - 1}$

$= \frac{\sin A(1 + 2\cos A)}{\cos A(1 + 2\cos A)}$

$= \frac{\sin A}{\cos A}$ = tan A = R.H.S

2) Prove that cos 4A = 8cos ^{4}A-8cos ^{2}A+1

Solution:

LHS = cos 2(2A)

= 2cos ^{2}2A-1

= 2(cos 2A)^2-1

= 2[2cos ^{2}A-1]^2-1

= 2[4cos ^{4}A+1-4cos ^{2}A]-1

= 8cos ^{4}A+2-8cos ^{2}A-1

= 8cos ^{4}A-8cos ^{2}A+1

= RHS

3) If tanα = $\frac{1}{3}$ and tan β = $\frac{1}{7}$ show that 2α+β=$\frac{\pi}{4}$

Solution:

Given: tan α = $\frac{1}{3}$, tan β = $\frac{1}{7}$

tan 2α = $\frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2\left(\frac{1}{3}\right)}{1 - \left(\frac{1}{3}\right)^2} = \frac{\frac{2}{3}}{1 - \frac{1}{9}}$

$$= \frac{2/3}{8/9} = \frac{2}{3} \cdot \frac{9}{8} = \frac{3}{4}$$

$\tan 2\alpha = \frac{3}{4}$

$$\tan(2\alpha+\beta) = \frac{\tan 2\alpha + \tan\beta}{1 - \tan 2\alpha \tan\beta}$$

$$= \frac{\frac{3}{4} + \frac{1}{7}}{1 - \left(\frac{3}{4}\right)\left(\frac{1}{7}\right)} = \frac{\frac{21+4}{28}}{1 - \frac{3}{28}}$$

$$= \frac{25/28}{25/28} = 1$$

$\tan(2\alpha+\beta) = 1$

$= \tan 45^\circ$

$2\alpha+\beta = 45^\circ$

$2\alpha+\beta = \frac{\pi}{4}$

4) If $\tan A = \frac{1-\cos B}{\sin B}$, prove that $\tan 2A = \tan B$ (A and B are acute angles)

Solution:

Given $\tan A = \frac{1-\cos B}{\sin B}$

$$= \frac{2\sin^2 B/2}{2\sin B/2 \cos B/2} = \frac{\sin B/2}{\cos B/2}$$

$\tan A = \tan B/2$

$\Rightarrow A = B/2$

$\Rightarrow 2A = B$

Taking 'tan' on both sides

$\tan 2A = \tan B$.

EXERCISE

PART A

1. Show that cos (-330^0) cos 420^0 = $\frac{\sqrt{3}}{4}$
2. Show that cos 780^0 sin 750^0= $\frac{1}{4}$
3. Find the value of the following: Sin (-330^0) cos 300^0
4. Cos (-390^0) sin 420^0
5. Cos $(90^0 + \theta)$ sec $(-\theta)$ tan $(180^0 - \theta)$
6. Cot $(90^0-\theta)$ sin $(180^0 + \theta)$ sec $(360^0-\theta)$
7. Sin 40^0 cos20^0 + cos 40^0 sin$20^{0\backslash}$
8. Sin 70^0 cos 10^0 – cos 70^0 sin 10^0
9. Cos 75^0 cos 15^0 + sin 75^0 sin 15^0
10. Cos 75^0 cos 15^0 – sin 75^0 sin 15^0
11. Find the value of sin 75^0
12. Find the value of cos 15^0
13. Prove that sin $(45^0 + A)$ = $\frac{1}{\sqrt{2}}$ (sin A + cos A)
14. Prove that cos $(A + 45^0)$ = $\frac{1}{\sqrt{2}}$ (cos A – sin A)
15. Prove that cos $(45^0 – A)$ cos $(45^0 – B)$ – sin $(45^0 – A)$ sin $(45^0 – B)$ = sin (A + B)
16. Prove that sin A + sin $(120^0 + A)$ – sin $(120^0 – A)$ = 0
17. Find that value of the following: $\frac{\tan 22^0 + \tan 23^0}{1 - \tan 22^0 \tan 23^0}$
18. $\frac{\tan 65^0 - \tan 20^0}{1 + \tan 65^0 \tan 20^0}$
19. If tan A = $\frac{5}{6}$ and tan B = $\frac{1}{11}$, find the value of tan (A + B)

20. Prove that tan $\left(\frac{\pi}{4}+\theta\right)$ tan $\left(\frac{\pi}{4}-\theta\right)$ = 1

21. Find the value of the following

    2 sin 15^0 cos 15^0

22. 1 – 2 $\sin^2 15^0$

23. 2 $\cos^2 30^0$ – 1

24. $\text{Cos}^2 15^0 - \sin^2 15^0$

25. $\dfrac{2\tan 22\frac{1}{2}^0}{1-\tan^2 22\frac{1}{2}^0}$

Prove the following:

26. $\dfrac{\sin 2A}{1-\cos 24} = \cot A$

27. $\dfrac{1-\tan^2(45-A)}{1+\tan^2(45-A)} = \sin^2 A$

28. $(\sin A + \cos A)^2 = 1 + \sin 2A$

29. $\cos^4 A - \sin^4 A = \cos 2A$

30. $\dfrac{\sin A}{1-\cos A} = \cot A/2$

PART –B

1. If sin A = $\frac{3}{5}$, Cos B = $\frac{12}{13}$, (A or B be acute) find

   (i) sin(A+B) and (ii) cos (A+B)

2. If sin A = $\frac{1}{3}$, sin B = $\frac{1}{4}$ (A&B are acute) find sin (A+B)

3. If sin A = $\frac{8}{17}$, sin B= $\frac{5}{13}$, (A&B are acute) prove that

   sin (A+B) = $\frac{171}{221}$

4. If sin A = $\frac{4}{5}$ cos B = $\frac{18}{17}$ (A&B be acute) find cos (A-B)

5. Prove that sin A + sin (120+A) + sin (240+A)= 0

6. Prove that cos A + cos (120+A) + cos (120-A)= 0

7. Prove that $\frac{\sin(A-B)}{\sin A \sin B}+\frac{\sin(B-C)}{\sin B \sin C}+\frac{\sin(C-A)}{\sin C \sin A}$ =0

8. Prove that $\frac{\sin(A-B)}{\cos A \cos B}+\frac{\sin(B-C)}{\cos B \cos C}+\frac{\sin(C-A)}{\cos C \cos A}$ =0

9. Prove that tan 20°+ tan 25°+ tan 20° tan 25°=1

10. If A+B=45°, prove that (cotA-1) (cot B-1)=2 and hence deduce the value of cot $22\frac{1^\circ}{2}$

11. If A+B =225° , prove that $\frac{\cot A \cot B}{(1+\cot A)(1+\cot B)}=\frac{1}{2}$

12. If A+B+C=180°,prove that cotA cotB+ cotB cotC+ cotC cot A=1

13. If A+B +C =180°,prove that tanA+ tanB+ tanC =tanA tanB tanC

14. prove that tan 5A- tan 3A- tan 2A= tan 5A tan 3Atan 2A

15. If tan $\alpha=\frac{1}{2}$ and tan $\beta\frac{1}{3}$ show that tan $(2\alpha+\beta)$=3

16. Prove that $\frac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A}$ = tan A

17. Prove that tan A +cot A= 2 cosec 2A

18. Prove that cot A -cot 2A = cosec 2A

19. Show that $\frac{\sin 2A}{1-\cos 2A}$ = cot A and hence deduce the value of cot 15° and cot $22\frac{1}{2}^\circ$

20. If tan $\theta=\frac{a}{b}$ find the value of a sin 2θ + b cos 2θ

21. Prove that cos 4A = 8sin ^{4}A- 8Sin ^{2}A+1

22. Prove that $\cos^6 A + \sin^6 A = 1 - \frac{3}{4}\sin^2 2A$

23. Prove that $\frac{1-\cos A+\sin A}{1+\cos A+\sin A} = \tan\frac{A}{2}$

24. If $\tan\alpha = \frac{1}{3}$ and $\tan\beta = \frac{1}{7}$, show that $2\alpha+\beta=45$

ANSWERS

PART –A

3) $\frac{1}{4}$ 4) $\frac{3}{4}$

5) $\sin\theta \sec\theta \tan\theta$ 6) $\tan\theta \sin\theta \sec\theta$

7) $\frac{\sqrt{3}}{2}$ 8) $\frac{\sqrt{3}}{2}$ 9) $\frac{1}{2}$ 10) 0

11) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ 12) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ 17) 1

18) 1 19) 1 21) $\frac{1}{2}$

22) $\frac{\sqrt{3}}{2}$ 23) $\frac{1}{2}$ 24) $\frac{\sqrt{3}}{2}$ 25) 1

Part B

1) i) $\frac{56}{65}$ ii) $\frac{33}{65}$

2) $\frac{\sqrt{15}+2\sqrt{2}}{12}$ 3) $\frac{77}{85}$

10) $\sqrt{2}+1$ 20) b

UNIT – V

TRIGONOMETRY

5.1 Trigonometrical ratios of multiple angels (3A only). Simple problems.

5.2 Sum and Product formulae-Simple Problems.

5.3 Definition of inverse trigonometric ratios, relation between inverse trigonometric ratios-Simple Problems.

5.1 TRIGONOMETRICAL RATIOS OF MULTIPLE ANGLE OF 3A.

i) $\sin 3A = \sin (A+2A)$

$= \sin A \cos 2A + \cos A \sin 2A$

$= \sin A (1-2 \sin^2 A) + \cos A (2 \sin A \cos A)$

$= \sin A - 2\sin^3 A + 2\sin A \cos^2 A$

$= \sin A - 2\sin^3 A + 2\sin A (1-\sin^2 A)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$

$\sin 3A = 3\sin A - 4\sin^3 A$

ii) $\text{Cos}3A = \text{Cos}(A+2A)$

$= \text{Cos}A\text{Cos}2A - \text{Sin}A \text{Sin}2A$

$= \text{Cos}A(2\text{Cos}^2 A - 1) - \text{Sin} A (2\text{Sin}A\text{Cos}A)$

$= 2\text{Cos}^3 A - \text{Cos}A - 2\text{Sin}^2 A \text{Cos} A$

$= 2\text{Cos}^3 A - \text{Cos}A - 2 (1-\text{Cos}^2 A) \text{Cos} A$

$= 2\text{Cos}^3 A - \text{Cos}A - 2\text{Cos}A + 2\text{Cos}^3 A$

$\text{Cos}3A = 4\text{Cos}^3 A - 3\text{Cos}A$

iii) $\tan 3A = \tan (A + 2A)$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$\tan 3A = \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A\left(\dfrac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{\tan A\,(1-\tan^2)\,A + 2\tan A}{1-\tan^2 A - 2\tan^2 A}$$

$$= \frac{\tan A - \tan^3 A + 2\tan A}{1 - 3\tan^2 A}$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

5.1 WORKED EXAMPLES

PART-A

1) Find the value of $3\sin 20^\circ - 4\sin^3 20^\circ$

Solution:

We have, $\sin 3A = 3\sin A - 4\sin^3 A$

$3\sin 20^0 - 4\sin^3 20^\circ = \sin 3\,(20^\circ)$ (Here $A = 20^\circ$)

$$= \sin 60^\circ$$

$$= \frac{\sqrt{3}}{2}$$

2) Find the value of $4\cos^3 10^\circ - 3\cos 10^\circ$

Solution:

We have, $\cos 3A = 4\cos^3 A - 3\cos A$

$\therefore 4\cos^3 10^\circ - 3\cos 10^\circ = \cos 3\,(10^\circ)$ (Here $A = 10^\circ$)

$$= \cos 30^\circ$$

$$= \frac{\sqrt{3}}{2}$$

3) If sin $\theta = \frac{3}{5}$, find the value of sin 3θ

Solution:

Given: $\sin\theta = \frac{3}{5}$

We have, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$= 3\left(\frac{3}{5}\right) - 4\left(\frac{3}{5}\right)^3$$

$$= \frac{9}{5} - \frac{108}{125} = \frac{225-108}{125}$$

$\text{Sin } 3\theta = \frac{117}{125}$

4) If cos A = $\frac{1}{3}$, find the value of cos3A

Solution:

Given : $\cos A = \frac{1}{3}$

We have $\cos 3A = 4\cos^3 A - 3\cos A$

$$= 4\left(\frac{1}{3}\right)^3 - 3\left(\frac{1}{3}\right)$$

$$= 4\left(\frac{1}{27}\right) - 1 = \frac{4-27}{27}$$

$\text{Cos } 3A = -\frac{23}{27}$

PART – B

1) Prove that $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$

Solution:

$$\text{LHS} = \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A}$$

$$= \frac{3\sin A - 4\sin^3 A}{\sin A} - \frac{4\cos^3 A - 3\cos A}{\cos A}$$

$$= \frac{\sin A(3 - 4\sin^2 A)}{\sin A} - \frac{\cos A(4\cos^2 A - 3)}{\cos A}$$

$= 3\text{-}4 \sin^2 A – 4\cos^2 A + 3$

$= 6 \text{ -}4 (\sin^2 A + \cos^2 A)$

$= 6 – 4$

$= 2$

= R.H.S

2) Prove that $\frac{\cos^3 A - \cos 3A}{\cos A} + \frac{\sin^3 A + \sin 3A}{\sin A} = 3$

Solution:

$$\text{LHS} = \frac{\cos^3 A - \cos 3A}{\cos A} + \frac{\sin^3 A + \sin 3A}{\sin A}$$

$$= \frac{\cos^3 A - (4\cos^3 A - 3\cos A)}{\cos A} + \frac{\sin^3 A + (3\sin A - 4\sin^3 A)}{\sin A}$$

$$= \frac{\cos^3 A - 4\cos^3 A + 3\cos A}{\cos A} + \frac{\sin^3 A + 3\sin A - 4\sin^3 A}{\sin A}$$

$$= \frac{3\cos A - 3\cos^3 A}{\cos A} + \frac{3\sin A - 3\sin^3 A}{\sin A}$$

$$= \frac{3\cos A(1 - \cos^2 A)}{\cos A} + \frac{3\sin A(1 - \sin^2 A)}{\sin A}$$

$= 3(1\text{-}\cos^2 A) + 3\ (1\text{-}\sin^2 A)$

$= 3 \sin^2 A + 3\cos^2 A$

$= 3(\sin^2 A + \cos^2 A)$

= 3(1) = 3 = R.H.S

3) Prove that $\frac{\sin 3A}{1+2\cos 2A}$ = sin A and hence find the value of sin 15°.

Solution:

LHS = $\frac{\sin 3A}{1+2\cos 2A}$

$= \frac{3\sin A - 4\sin^3 A}{1+2(1-2\sin^2 A)}$

$= \frac{\sin A(3-4\sin 2A)}{1+2-4\sin^2 A)}$

$= \frac{\sin A(3-4\sin^2 A)}{3-4\sin^2 A}$

= sin A = RHS

We have proved that sin A = $\frac{\sin 3A}{1+2\cos 2A}$

put A = 15°

$\therefore \sin 15° = \frac{\sin 3(15°)}{1+2\cos 2(15°)} = \frac{\sin 45°}{1+2\cos 30°}$

$= \frac{\frac{1}{\sqrt{2}}}{1+2(\frac{\sqrt{3}}{2})} = \frac{\frac{1}{\sqrt{2}}}{1+\sqrt{3}}$

$= \frac{1}{\sqrt{2}(1+\sqrt{3})}$

4) Prove that 4sinA sin (60+A) sin (60-A) = sin3A

Solution:

LHS = 4sin A sin (60+A) sin (60-A)

= 4sin A [$\sin^2 60 - \sin^2 A$]   [sin (A+B) sin (A-B) = $\sin^2 A - \sin^2 B$]

$= 4\sin A\ [(\frac{\sqrt{3}}{2})^2 - \sin^2 A]$

$= 4\sin A\ [\frac{3}{4} - \sin^2 A]$

$= 4\sin A \left[\frac{3 - 4\sin^2 A}{4}\right]$

$= 3\sin A - 4\sin^3 A$

$= \sin 3A$ = RHS

5) Prove that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$

Solution:

LHS = $\cos 20^\circ \cos 40^\circ \cos 80^\circ$

$= \cos 20^\circ \cos(60^\circ - 20^\circ) \cos(60^\circ + 20^\circ)$

$= \cos 20^\circ [\cos^2 60^\circ - \sin^2 20^\circ]$ $[\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B]$

$= \cos 20^\circ \left[\left(\frac{1}{2}\right)^2 - \left(1 - \cos^2 20^\circ\right)\right]$

$= \cos 20^\circ \left[\frac{1}{4} - 1 + \cos^2 20^\circ\right]$

$= \cos 20^\circ \left[\frac{4\cos^2 20^\circ - 3}{4}\right]$

$= \frac{1}{4}\ [4\cos^3 20^\circ - 3\cos 20^\circ]$

$= \frac{1}{4}\ \cos 3(20^\circ)$

$= \frac{1}{4}\ \cos 60^\circ = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$ =RHS

5.2 - SUM AND PRODUCT FORMULAE

Sum or Difference formulae:

We know that

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ (1)

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ (2)

Adding (1) and (2), we get

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$ (I)

Subtracting (2) from (1), we get

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$ (II)

We know that

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ (3)

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ (4)

Adding (3) and (4), we get

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$ (III)

Subtracting (4) from (3), we get

$\cos(A+B) - \cos(A-B) = -2\sin A \sin B$ (IV)

(or)

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

$\mathbf{\sin(A+B) + \sin(A-B) = 2\sin A \cos B}$

$\mathbf{\sin(A+B) - \sin(A-B) = 2\cos A \sin B}$

$\mathbf{\cos(A+B) + \cos(A-B) = 2\cos A \cos B}$

$\mathbf{\cos(A+B) - \cos(A-B) = -2\sin A \sin B}$

(or)

$\mathbf{\cos(A-B) - \cos(A+B) = 2\sin A \sin B}$

Product formula:

Let $C = A + B$ and $D = A - B$

Then $C + D = 2A$ and $C - D = 2B$

$\Rightarrow A = \frac{C+D}{2} \quad \Rightarrow B = \frac{C-D}{2}$

Putting these values of A and B in **I**, **II**, **III**, IV we get

$$\mathbf{\sin C + \sin D = 2\sin} \frac{C+D}{2} \mathbf{\cos} \frac{C-D}{2}$$

$$\mathbf{\sin C - \sin D = 2\cos} \frac{C+D}{2} \mathbf{\sin} \frac{C-D}{2}$$

$$\mathbf{\cos C + \cos D = 2\cos} \frac{C+D}{2} \mathbf{\cos} \frac{C-D}{2}$$

$$\mathbf{\cos C - \cos D = -2\sin} \frac{C+D}{2} \mathbf{\sin} \frac{C-D}{2}$$

(or)

$$\mathbf{\cos D - \cos C = 2\sin} \frac{C+D}{2} \mathbf{\sin} \frac{C-D}{2}$$

5.2 WORKED EXAMPLES

PART – A

1) Express the following as sum or difference:

i) $2\sin2\theta\cos\theta$ ii) $2\cos3A\sin5A$

iii) $2\cos3A\cos2A$ iv) $2\sin3A\sin A$

Solution:

i) We have $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

$2\sin2\theta\cos\theta = \sin(2\theta + \theta) + \sin(2\theta - \theta)$

$= \sin3\theta + \sin\theta$

ii) We have $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

$2\cos3A \sin5A = \sin(3A + 5A) - \sin(3A - 5A)$

$= \sin8A - \sin(-2A)$

$= \sin8A + \sin2A$

$[\therefore \sin(-A) = -\sin A]$

iii) We have, $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

$2\cos 3A \cos 2A = \cos(3A + 2A) + \cos(3A - 2A)$

$= \cos 5A + \cos A$

iv) We have $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$2\sin 3A \sin A = \cos(3A-A) - \cos(3A+A)$

$= \cos 2A - \cos 4A.$

2) Express the following as product:

1. $\sin 4A + \sin 2A$ 2. $\sin 5A - \sin 3A$

3. $\cos 3A + \cos 7A$ 4. $\cos 2A - \cos 4A$

Solution:

i) We have $\sin C + \sin D = 2\sin \frac{C+D}{2} \cos \frac{C-D}{2}$

$\sin 4A + \sin 2A = 2\sin \frac{4A+2A}{2} \cos \frac{4A-2A}{2}$

$= 2\sin 3A \cos A$

We have $\sin C - \sin D = 2\cos \frac{C+D}{2} \sin \frac{C-D}{2}$

ii) $\sin 5A - \sin 3A = 2\cos \frac{5A+3A}{2} \sin \frac{5A-3A}{2}$

$= 2\cos 4A \sin A$

We have $\cos C + \cos D = 2\cos \frac{C+D}{2} \cos \frac{C-D}{2}$

iii) $\cos 3A + \cos 7A = 2\cos \frac{3A+7A}{2} \cos \frac{3A-7A}{2}$

$= 2\cos 5A \cos(-2A)$

$= 2\cos 5A \cos 2A \left(\because \cos(-\theta) = \cos\theta\right)$

iv) We Have $\cos C - \cos D = -2\sin \frac{C+D}{2} \sin \frac{C-D}{2}$

$\cos 2A - \cos 4A = -2 \sin \frac{2A+4A}{2} \sin \frac{2A-4A}{2}$

$= -2 \sin 3A \sin (-A)$

$= -2 \sin 3A \sin A$

3) Prove that $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$

Solution:

LHS $= \sin 10^\circ + \sin 50^\circ - \sin 70^\circ$

$= \sin 10^\circ + 2\cos \left(\frac{50^\circ + 70^\circ}{2}\right) \sin \left(\frac{50^\circ - 70^\circ}{2}\right)$

$= \sin 10^\circ + 2\cos 60^\circ \sin (-10^\circ)$

$= \sin 10^\circ + 2 (\frac{1}{2}) (-\sin 10^\circ)$

$= \sin 10^\circ - \sin 10^\circ$

$= 0 =$ RHS

4) Prove that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

Solution:

LHS $= \cos 20^\circ + \cos 100^\circ + \cos 140^\circ$

$= \cos 20^\circ + 2\cos \left(\frac{100^\circ + 140^\circ}{2}\right) \cos \left(\frac{100^\circ - 140^\circ}{2}\right)$

$= \cos 20^\circ + 2\cos 120^\circ \cos (-20^\circ)$

$= \cos 20^\circ + 2 \left(\frac{-1}{2}\right) \cos 20^\circ$

$= \cos 20^\circ - \cos 20^\circ$

$= 0 =$ RHS.

5) Prove that cos A + cos (120° + A) + cos (120°-A) = 0

Solution:

LHS = cos A+ cos (120°+A) + cos (120°-A)

$$= \cos A + 2\cos\left[\frac{120^\circ + A + 120^\circ - A}{2}\right]\cos\left[\frac{120^\circ + A - 120^\circ + A}{2}\right]$$

$$= \cos A + 2\cos 120^\circ \cos A$$

$$= \cos A + 2\left(\frac{-1}{2}\right)\cos A$$

$$= \cos A - \cos A$$

= 0 = RHS.

6) Prove that $\dfrac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B}$ = tan (A+B)

Solution:

$$\text{LHS} = \frac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B}$$

$$= \frac{2\cos\frac{(2A+2B)}{2}\sin(\frac{2A-2B)}{2}}{2\cos(\frac{2A+2B}{2})\cos(\frac{2A-2B}{2})}$$

$$= \frac{\sin(A-B)}{\cos(A-B)}$$

= tan (A-B) = RHS.

PART-B

1) Prove that (cos α + cos β) ^{2} + (sin α + sin β)^2 = 4cos^2 $\left(\dfrac{\alpha-\beta}{2}\right)$

Solution:

LHS = (cos α + cos β) ^{2} + (sin α + sin β)^2

$$= \left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2 + \left[2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2$$

$$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right) + 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left[\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)(1)$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

= RHS.

2) If sin x + sin y = a and cos x + cos y =b,

Prove that $\tan^2\left(\frac{x-y}{2}\right) = \frac{4 - a^2 - b^2}{a^2 + b^2}$

Solution:

Given: sin x + sin y = a and cos x + cos y =b

Consider $a^2+b^2 = (\sin x + \sin y)^2 + (\cos x + \cos y)^2$

$$= \left[2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\right]^2 + \left[2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\right]^2$$

$$= 4\sin^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) + 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right)$$

$$= 4\cos^2\left(\frac{x-y}{2}\right)\left[\sin^2\left(\frac{x+y}{2}\right) + \cos^2\left(\frac{x+y}{2}\right)\right]$$

$$= 4\cos^2\left(\frac{x-y}{2}\right)(1) \text{ since } [\sin^2\theta + \cos^2\theta = 1]$$

$$= 4\cos^2\left(\frac{x-y}{2}\right)$$

$$\text{R.H.S} = \frac{4 - a^2 - b^2}{a^2 + b^2} = \frac{4 - (a^2 + b^2)}{a^2 + b^2}$$

$$= \frac{4 - 4\cos^2(\frac{x-y}{2})}{4\cos^2(\frac{x-y}{2})}$$

$$= \frac{4(1 - \cos^2(\frac{x-y}{2})}{4\cos^2(\frac{x-y}{2})}$$

$$= \frac{\sin^2(\frac{x-y}{2})}{\cos^2(\frac{x-y}{2})} = \tan^2(\frac{x-y}{2}) = L.H.S$$

3) If sin x + sin y = a and cos x + cos y =b

Prove that $\sin(x+y) = \frac{2ab}{a^2+b^2}$

Solution:

Consider 2ab = 2 (sin x + sin y) (cos x + cos y)

$$= 2\,(2\sin\frac{x+y}{2}\,\cos\frac{x-y}{2})\,(2\cos\frac{x+y}{2}\,\cos\frac{x-y}{2})$$

$$= 4.(2)\sin\frac{x+y}{2}\,\cos\frac{x+y}{2}\,\cos^2\frac{x-y}{2}$$

$$= 4\cos^2\frac{x-y}{2},\ 2\sin\frac{x+y}{2}\,\cos\frac{x+y}{2}$$

$$= 4\cos^2\frac{x-y}{2},\ \sin 2\left(\frac{x+y}{2}\right)$$

[$\therefore$2 sin A cos A = sin2A]

$$= 4\cos^2\frac{x-y}{2}\,[\sin(x+y)]$$

Again $a^2+b^2 = 4\cos^2\frac{x-y}{2}$ (refer example 2)

$$\text{R.H.S } \frac{2ab}{a^2+b^2} = \frac{\left(4\cos^2\frac{x-y}{2}\right)\sin(x+y)}{4\cos^2\frac{x-y}{2}}$$

= sin (x+y)=L.H.S

4) Prove that $\frac{\sin A+\sin 2A+\sin 3A}{\cos A+\cos 2A+\cos 3A}$ = tan 2A

$$\text{LHS} = \frac{\sin 2A+(\sin 3A+\sin A)}{\cos 2A+(\cos 3A+\cos A)}$$

$$= \frac{\sin 2A+2\sin\frac{3A+A}{2}\cos\frac{3A-A}{2}}{\cos 2A+2\cos\frac{3A+A}{2}\cos\frac{3A-A}{2}}$$

$$= \frac{\sin 2A+2\sin 2A\cos A}{\cos 2A+2\cos 2A\cos A}$$

$$= \frac{\sin 2A(1+2\cos A)}{\cos 2A(1+2\cos A)}$$

$$= \frac{\sin 2A}{\cos 2A}$$

= tan 2A

= RHS.

5.3 INVERSE TRIGONOMETRIC FUNCTION

INVERSE TRIGONOMETRIC FUNCTION :

Definition: A Function which does the reverse process of a trigonometric function is called inverse trigonometric function.

The domain of trigonometric function is set of angles and range is set of real members.

In case of inverse trigonometric function the domain is set of real numbers and range is set of angles.

Inverse trigonometric function of "sine" is denoted as $\sin^{-1}$ similarly $\cos^{-1}$, $\tan^{-1}$, $\cos^{-1}$, $\sec^{-1}$ and cosec^{-1} are inverse trigonometric functions of cos, tan, cot, sec and cosec functions respectively.

Examples:

We know $\sin 30° = \frac{1}{2} \therefore \text{Sin}^{-1}\left(\frac{1}{2}\right) = 30°$

$\tan 45° = 1 \therefore \tan^{-1}(1) = 45°$

$\cos 0° = 1 \therefore \cos^{-1}(1) = 0°$

Note:

i) There is a difference between $\sin^{-1}x$ and $(\sin x)^{-1}$ $\sin^{-1}x$ is inverse trigonometric function of sin x whereas $(\sin x)^{-1}$ is the reciprocal of sin x. i.e. $(\sin x)^{-1} = \frac{1}{\sin x}$ = cosec x.

ii) Value of inverse trigonometric function is an angle.

i.e. $\sin^{-1}x$ is an a angle

i.e. $\sin^{-1}x = \theta$

Principal value:

Among all the values, the numerically least value of the inverse trigonometric function is called principal value.

Examples:

1) We know $\sin 30° = \frac{1}{2}$, $\sin 150° = \frac{1}{2}$

$\text{Sin } 390° = \frac{1}{2}$, $-\sin(-330°) = \frac{1}{2}$ .....

$\text{Sin}^{-1}\left(\frac{1}{2}\right) = 30°, 150°, 390°, -330°$

The least positive value is 30°, which is called principal value of

$\sin^{-1}\left(\frac{1}{2}\right)$

2) We know $\cos 60° = \frac{1}{2}$, $\cos(-60°) = \frac{1}{2}$

$\text{Cos } 300° = \frac{1}{2}$, $\cos 420° = \frac{1}{2}$ ......

$\text{Cos}^{-1}\left(\frac{1}{2}\right) = 60°, -60°, 300°, 420°.......$

The principal value of $\cos^{-1}\left(\frac{1}{2}\right)$ is 60°

Here 60° and -60° are numerically equal and though - 60° is the smaller than 60° only 60° is taken as principal value.

The following table gives the range of principal values of the inverse trigonometric functions.

Function	Domain	Range of Principal Value of θ
Sin^{-1} x	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
Cos^{-1} x	$-1 \le x \le 1$	$0 \le \theta \le \frac{\pi}{2}$
Tan^{-1} x	$(-\infty,\infty)$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
Cot^{-1} x	$(-\infty,\infty)$	$0 < \theta < \pi$
Sec^{-1} x	(-1,1) except o	$0 < \theta < \pi, \theta \neq \frac{\pi}{2}$
Cosec^{-1} x	(-1,1) except o	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \theta \neq 0°$

Properties

Property (1)

(a) $\sin^{-1}(\sin x) = x$

(b) $\cos^{-1}(\cos x) = x$

(c) $\tan^{-1}(\tan x) = x$

(d) $\cot^{-1}(\cot x) = x$

(e) $\sec^{-1}(\sec x) = x$

(f) $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$

Proof:

(a) let $\sin x = y \quad \rightarrow x = \sin^{-1}(y)$ (1)

Put $\sin x = y$ in $\sin^{-1}(\sin x)$

$\therefore \ \text{Sin}^{-1}(\sin x) = \sin^{-1}(y)$ (2)

From (1) and (2), $\sin^{-1}(\sin x) = x$

Similarly other results can be proved

Property (2)

(a) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$

(b) $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x$

(c) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$

(d) $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x$

(e) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$

(f) $\cot^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right] = \tan^{-1}x$

Proof:

(a) Let $\sin^{-1}\left(\frac{1}{x}\right) = y$ -------------- 1

$$\text{Sin } y = \frac{1}{x}$$

$$\therefore x = \frac{1}{\sin y} = \text{cosec } y$$

$$\therefore y = \text{cosec}^{-1} x \qquad \text{-------------------} \quad 2$$

(1) and (2) $\Rightarrow \sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$

Similarly other results can be proved

Property (3)

(a) $\sin^{-1}(-x) = -\sin^{-1} x$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(c) $\tan^{-1}(-x) = -\tan^{-1} x$

(d) $\cot^{-1}(-x) = -\cot^{-1} x$

(e) $\sec^{-1}(-x) = -\sec^{-1} x$

(f) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

Proof:

(a) Let $y = \sin^{-1}(-x)$ (1)

$\sin y = -x$

i.e. $x = -\sin y$

$x = \sin(-y)$

i.e. $\sin^{-1} x = -y$

$\therefore y = -\sin^{-1} x$ (2)

From (1) and (2) $\sin^{-1}(-x) = -\sin^{-1} x$

(b) Let $y = \cos^{-1}(-x)$ (1)

$\text{Cos } y = -x$

i.e. $x = -\cos y$

$x = \cos(180° - y)$

$\cos^{-1} x = 180° - y$

$y = 180° - \cos^{-1} x$

$y = \pi - \cos^{-1} x$ (2)

From (1) and (2) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Similarly other results can be proved

Property (4)

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$

Proof:

(i) Let $\theta = \sin^{-1} x$ ………. (1)

$\sin\theta = x$

i.e. $x = \sin\theta$

$x = \cos(90° - \theta)$

$\therefore \cos^{-1} x = 90° - \theta$

$\cos^{-1} x = 90° - \sin^{-1} x$ (using (1))

$\sin^{-1} x + \cos^{-1} x = 90° = \frac{\pi}{2}$

Similarly other results can be proved

Property (5)

If $xy < 1$ Prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Let $A = \tan^{-1} x$ $\therefore x = \tan A$

Let $B = \tan^{-1} y$ $\therefore y = \tan B$

We know $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\therefore \tan(A+B) = \frac{x+y}{1-xy}$$

$$\therefore A + B = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\text{i.e } \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Property (6)

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\ \right]$$

Proof:

Let $A = \sin^{-1} x \ \therefore\ x = \sin A$

Let $B = \sin^{-1} y \ \therefore\ y = \sin B$

We know $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \sin A \sqrt{1-\sin^2 B} + \sqrt{1-\sin^2 A}\ \sin B$$

$$= x\sqrt{1-y^2} + \sqrt{1-x^2}\ y$$

$$A+B = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\text{i.e } \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

5.3 WORKED EXAMPLES

PART-A

1) Find the principal value of

(i) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

(iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (iv) $\cos^{-1}\left(-\frac{1}{2}\right)$

(v) $\operatorname{cosec}^{-1}(-2)$

Solution:

i) Let x = $\cos^{-1} \dfrac{\sqrt{3}}{2}$

$\therefore$ cos x = $\dfrac{\sqrt{3}}{2}$

cos x = cos 30°

$\therefore$ x = 30°

$\therefore \cos^{-1} \dfrac{\sqrt{3}}{2} = 30°$

ii) Let x = $\sin^{-1} \dfrac{1}{\sqrt{2}}$

$\therefore$ sin x = $\dfrac{1}{\sqrt{2}}$

sin x= sin 45°

$\therefore$ x= 45°

$\therefore \sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 45°$

iii) Let x = $\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

$\therefore$ tan x = $-\dfrac{1}{3}$

tan x =-tan 30°

tan x = tan (-30°)

$\therefore$ x = -30°

iv) Let x = $\cos^{-1}\left(-\dfrac{1}{2}\right)$

$\therefore$ cos x = $-\dfrac{1}{2}$

cos x = -cos 60°

$\cos x = \cos(180^\circ - 60^\circ) = \cos 120^\circ$

$\therefore\ x = 120^\circ$

v. Let $x = \operatorname{cosec}^{-1}(-2)$

$\therefore\ \operatorname{cosec} x = -2$

$\operatorname{cosec} x = -\operatorname{cosec} 30^\circ$

$\operatorname{cosec} x = \operatorname{cosec}(-30^\circ)$

$\therefore\ x = -30^\circ$

$\therefore \operatorname{cosec}^{-1}(-2) = -30^\circ$

2) Prove that $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{2}{5}\right) = \tan^{-1}\left(\frac{11}{13}\right)$

Solution:

We know $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\therefore \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{2}{5}\right) = \tan^{-1}\left[\frac{\frac{1}{3}+\frac{2}{5}}{1-\left(\frac{1}{3}\right)\left(\frac{2}{5}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{5+6}{15}}{\frac{15-2}{15}}\right]$$

$$= \tan^{-1}\left(\frac{11}{13}\right)$$

$$= \text{RHS}$$

3) Show that $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

Solution:

LHS $= \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$

$= \tan^{-1} x + \cot^{-1} x$ using properity (2)

$= \frac{\pi}{2}$ using properity (4)

PART-B

1) Show that $2\tan^{-1}x = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]$

Solution:

Let $x = \tan\theta \therefore \theta = \tan^{-1}x$

$$\text{LHS} = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]$$

$$= \cos^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$= \cos^{-1}[\cos 2\theta]$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

$$= \text{LHS}$$

2) Show that $\tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] = 3\tan^{-1}x$

Solution:

Let $= \tan\theta \quad \therefore \theta = \tan^{-1}x$

$$\text{LHS} = \tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right]$$

$$= \tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]$$

$$= \tan^{-1}[\tan 3\theta]$$

$$= 3\theta$$

$$= 3\tan^{-1}x = \text{R.H.S}$$

3) Show that $2\tan^{-1}\left(\frac{2}{3}\right) = \tan^{-1}\left(\frac{12}{15}\right)$

Solution:

$$\text{LHS} = 2\tan^{-1}\left(\frac{2}{3}\right)$$

$$= \tan^{-1}\left(\frac{2}{3}\right) + \tan^{-1}\left(\frac{2}{3}\right)$$

$$= \tan^{-1}\left[\frac{\frac{2}{3}+\frac{2}{3}}{1-(\frac{2}{3})(\frac{2}{3})}\right] \qquad \text{using properity (5)}$$

$$= \tan^{-1}\left[\frac{\frac{4}{3}}{\frac{9-4}{9}}\right]$$

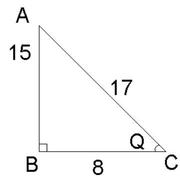
$$= \tan^{-1}\left(\frac{4}{3}\times\frac{9}{5}\right)$$

$$= \tan^{-1}\left(\frac{12}{15}\right) = \text{R.H.S}$$

4) Evaluate $\tan\left[\cos^{-1}\left(\frac{8}{17}\right)\right]$

Solution:

Let $\cos^{-1}\left(\frac{8}{17}\right) = \theta \qquad (1)$



$\text{Cos}\ \theta = \frac{8}{17}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{AB}{BC}$

$= \frac{15}{8}$

$\therefore \quad \theta = \tan^{-1} \frac{15}{8} \qquad (2)$

$$AB = \sqrt{17^2 - 8^2}$$
$$= \sqrt{289 - 64}$$
$$= \sqrt{225}$$
$$= 15$$

From (1) and (2) $\cos^{-1}\left(\frac{8}{17}\right) = \tan^{-1}\left(\frac{15}{8}\right)$

Taking 'tan ' on both sides

$$\tan\left[\cos^{-1}\left(\frac{8}{17}\right)\right] = \tan\left[\tan^{-1}\frac{15}{8}\right] = \frac{15}{8}$$

EXERCISE

PART-A

Find the value of the following:

1. $3\sin 10° - 4\sin^3 10°$
2. $4\sin^3 20° - 3\cos 20°$
3. $\dfrac{3\tan 20° - \tan^3 20°}{1 - 3\tan^2 20°}$
4. If $\sin A = \frac{4}{5}$ (A being acute) find sin3A
5. If $\cos\theta = \frac{3}{5}$ (θbeing acute) find $\cos 3\theta$
6. If $\tan\theta = 3$, find $\tan 3\theta$
7. Express in the form of a sum or difference.
   I. $2\sin 4\theta \cos 2\theta$
   II. $2\cos 8\theta \cos 6\theta$

III. $2\cos 6A \sin 3A$

IV. $2\sin 6\theta \sin 2\theta$

V. $\cos \frac{3A}{2} \sin \frac{A}{2}$

VI. $\sin \frac{7A}{2} \cos \frac{5A}{2}$

VII. $\cos (60+\alpha) \sin (60- \alpha)$

8. Express is the form of a product.

I. $\sin 13 A + \sin 5A$

II. $\sin 13 A - \sin 5A$

III. $\cos 13 A + \cos 5A$

IV. $\cos 13 A - \cos 5A$

V. $\sin 52^o – \sin 32^o$

VI. $\sin 50^o + \cos 80^o$

VII. $\sin 20^o + \cos 50^o$

VIII. $\cos 35^0 + \sin 72^0$

Prove the following :

9. $\sin 10^o + \sin 50^o – \sin 70^o = 0$

10. $\sin 20^o + \sin 40^0 – \sin 80^o = 0$

11. $\sin 78^o – \sin 18^o + \cos 132^o = 0$

12. $\text{Cos } 80^o + \cos 40^o – \cos 20^o = 0$

13. $\text{Cos } 52^o + \cos 68^o + \cos 172^o = 0$

14. $\text{Cos } 40^o + \cos 80^o + \cos 160^o = 0$

Prove the following:

15. $\text{Sin } 50^o + \sin 10^o = \cos 20^o$

16. $\text{Cos } 10^o + \cos 70^o = \sqrt{3} \cos 40^o$

17. $\text{Sin } A + \sin (120^0+A) – \sin (120^o-A) = 0$

18. Sin A + sin (120^0+A) + sin $(240^o+A) = 0$

19. Cos A + cos (240^o-A) + cos $(240^o+A) = 0$

20. $\frac{\sin 3A - \sin A}{\cos 3A - \cos A} = \cot 2A$

21. $\frac{\sin 7A - \sin 5A}{\cos 7A + \cos 5A} = \tan A$

22. $\frac{\sin 3A + \sin A}{\cos 3A + \sin A} = \tan 2A$

23. $\frac{\cos B - \cos A}{\sin A - \sin B} = \tan\left(\frac{A+B}{2}\right)$

24. Find the principal value of

i) $\text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right)$

ii) $\text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$

iii) $\text{Sin}^{-1}(-1)$

iv) $\text{Sin}^{-1}\left(-\frac{1}{2}\right)$

v) $\text{Cos}^{-1}(0)$

vi) $\text{Cos}^{-1}\left(\frac{1}{\sqrt{2}}\right)$

vii) $\tan^{-1}\left(\sqrt{3}\right)$

viii) $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

ix) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Prove the following:

25. $\text{Sin}^{-1}(x) + \sin^{-1}\frac{1}{x} = \frac{\Pi}{2}$

26. $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}(1)$

27. $\text{Sin}^{-1}\left(\sqrt{1-x^2}\right) = \cos^{-1} x$

28. $\text{Sec}^{-1}\left(\sqrt{1+x^2}\right) = \tan^{-1} x$

PART – B

Prove the following:

1) $\frac{\sin 3\theta}{\sin\theta} + \frac{\cos 3\theta}{\cos\theta} = 4\cos 2\theta$

2) $\frac{\cos 3A - \sin 3A}{\cos A + \sin A} = 1\text{-}2\sin 2A$

3) $\frac{\sin 3A + \sin^3 A}{\cos^3 A - \cos 3A} = \cot A$

4) Prove that $\frac{1-\cos 3A}{1-\cos A} = (1+2\cos A)^2$

5) Prove that $\frac{\cos 3A}{2\cos 2A - 1} = \cos A$ and hence

   deduce the value $\cos 15^0$

6) Prove that $4(\cos^3 10^o + \sin^3 20^o) = 3(\cos 10^o + \sin 20^o)$

7) Prove that $4\sin A\sin(60^o + A)\sin(120^o + A) = \sin 3A$

8) Prove that $\sin 20^o \sin 40^o \sin 80^o = \frac{\sqrt{3}}{8}$

9) Prove that $\sin 10^o \sin 50^o \sin 70^o = \frac{1}{8}$

10) Prove that $4 \cos \theta \cos (60^o - \theta) \cos (60^o + \theta) = \cos 3\theta$

11) Prove that $\sin 10^o \sin 30^o \sin 50^o \sin 70^o = \frac{1}{16}$

12) Prove that $\sin 20^o \sin 40^o \sin 60^o \sin 80^o = \frac{3}{16}$

13) Prove that $\text{Cos } 10^o \cos 30^o \cos 50^o \cos 70^o = \frac{3}{16}$

14) Prove that $\cos 20^o \cos 40^o \cos 60^o \cos 80^o = \frac{1}{16}$

15) Prove that $\tan\theta \tan (60^o + \theta) \tan (60^o - \theta) = \tan 3\theta$

16) Prove that $\tan 20^o \tan 40^o \tan 80^o = \sqrt{3}$

17) Prove that $\frac{\sin 2A + \sin 5A - \sin A}{\cos 2A + \cos 5A + \cos A} = \tan 2A$

18) Prove that $\frac{\cos 2A + \cos 5A + \cos A}{\sin 2A + \sin 5A - \sin A} = \cot 2A$

19) Prove that $\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

20) Prove that $(\text{Cos } \alpha + \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \cos^2 \frac{\alpha + \beta}{2}$

21) Prove that $(\text{Cos } \alpha - \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \sin^2 \frac{\alpha + \beta}{2}$

22) Prove that $(\text{Cos } \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \sin^2 \frac{\alpha - \beta}{2}$

23) If $\sin x + \sin y = a$ and $\cos x + \cos y = b$, prove that

$$\sec^2 \left(\frac{x - y}{2}\right) = \frac{4}{a^2 + b^2}$$

24) If sinx - sin y = a and cos x - cos y = b , prove that

$$\sec^2\left(\frac{x-y}{2}\right)=\frac{4}{4-a^2-b^2}$$

25) If sin x-sin y = a and cos x – cos y = b, prove that

$$\tan^2\left(\frac{x-y}{2}\right)=\frac{a^2+b^2}{4-a^2-b^2}$$

26) Prove that $\sin^2\theta + \sin^2(120+0) + \sin^2(120-0) = \frac{3}{2}$

27) Prove that $\cos^2 A+\cos^2(A+120) + \cos^2(A-120) = \frac{3}{2}$

Prove the following:

28) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

29) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

30) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

31) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2}-y\sqrt{1-x^2}\right]$

32) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left[xy-\sqrt{1-x^2}\sqrt{1-y^2}\right]$

33) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left[xy+\sqrt{1-x^2}\sqrt{1-y^2}\right]$

34) $\tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}=\frac{x}{2}\right]$

35) $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

36) Evaluate $\cos\left[\sin^{-1}\left(\frac{5}{13}\right)\right]$

37) Evaluate $\cos\left[\sin^{-1}\frac{4}{5}+\sin^{-1}\left(\frac{12}{13}\right)\right]$

ANSWERS

PART-A

1) $\frac{1}{2}$ 2) $\frac{1}{2}$ 3) $\sqrt{3}$ 4) $\frac{44}{125}$ 5) $\frac{-117}{125}$

6) $\frac{9}{13}$

7) (i) $\sin 6\theta + \sin 2\theta$

(ii) $\cos 14\,\theta + \cos 2\,\theta$

(iii) $\sin 9\,A - \sin 3\,A$

(iv) $\cos 4\theta - \cos 8\,\theta$

(v) $\frac{1}{2}$ [sin2A – sinA]

(vi) $\frac{1}{2}$ [sin 6 A + sin A]

(vii) $\frac{1}{2}\left[\frac{\sqrt{3}}{2}-\sin 2\alpha\right]$

8) (i) 2sin 9A cos 4A

(ii) 2cos 9A sin 4A

(iii) 2cos 9A sin 4A

(iv) -2sin 9A sin 4A

(v) $2\cos 42^0 \sin 10^o$

(vi) $2\sin 30^0 \cos 20^0$

(vii) $2\sin 30^0 \cos 20^0$

(viii) $2\cos\frac{53^\circ}{2}\cos\frac{17^\circ}{2}$

24) (i) $\frac{\pi}{4}$ ii) $\frac{\pi}{3}$ iii) $\frac{-\pi}{2}$ iv) $\frac{-\pi}{3}$ v) $\frac{\pi}{2}$

vi) $\frac{3\pi}{4}$ vii) $\frac{\pi}{3}$ viii) $\frac{\pi}{6}$ ix) $\frac{-\pi}{4}$

PART – B

35) $\frac{1}{\sqrt{6}-\sqrt{2}}$ 36) $\frac{12}{13}$ 37) $\frac{-33}{65}$

MATHEMATICS – I

MODEL QUESTION PAPER - 1

Time : 3 Hrs (Maximum Marks: 75)

PART – A

(Marks: 15 x 1 = 15)

I. Answer Any 15 Questions

1. Solve $\begin{vmatrix} x & 2x \\ 3 & 2x \end{vmatrix} = 0$

2. Find the Value $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = 0$

3. If A = $\begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 \\ 7 & -2 \end{pmatrix}$ FindAB

4. Find the adjoint Matrix of $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$

5. Find the value of $10C_7$

6. Find the general term of $(3x - y)^8$

7. Expand $(1 + x)^{-3}$upto three terms when $|x| < 1$

8. Split $\frac{x+1}{x(x+1)}$ into partial fraction without finding the constant

9. Find the value of 'm' if the lines 2x+my = 4 and x + 5y-6=0 are perpendicular

10. Find the combined the equation of the lines 2x + 5y = 0 and x + 3y = 0

11. Show that the pair of lines x^2 - 8y + $16y^2$ = 0 are parallel

12. Write down the condition for the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines.

13. Show that sin (-330)° x sin 420° = $\sqrt{\frac{3}{4}}$

14. Find the value of cos50° cos40° - sin50° sin40°

15. If $\tan A = \frac{1}{2}$ $\tan B = \frac{1}{3}$ Find tan (A + B)

16. Find the value of 2 sin 75° cos75°

17. If sinθ = 1/3, findthe value of sin 3 θ

18. Find the value of $4\cos^3 10°$ - 3cos 10°

19. Showthat $\frac{\sin 2A - \sin 2B}{\cos 2A - \cos 2B}$ = -cot (A+B)

20. Show that $\tan^{-1} \frac{2x}{1-x^2} = 2\tan^{-1} x$

PART – B

(Answer Any TWO subdivisions in each question)

All Questions carry Equal Marks

5x12=60

21a. Solve by using Cramer's Rule

x+y+z=3, 2x-y+z=2 and 3x+2y-2z=3

b Short that $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix} = x^2(x+3)$

c. Find the inverse of $\begin{vmatrix} 3 & -2 & 1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix}$

22.a Find the middle terms in the expansion of $\left(x^3 + \frac{2}{x^3}\right)^{11}$

b Find the term independent of x in the expansion of $(2x^2+1/x)^{12}$

c. Resolve $\frac{x-3}{x(x+5)(x-6)}$ in to partial fraction

23.a. Find the angle between the lines 7x+ 2y = 1 andx-5y=0

b. Find the separate equation of the pair of straight lines $9x^2+12xy+4y^2=0$. Also prove that the lines are parallel.

c Show that the equations represented by$2x^2 - 7xy + 3y^2 + 5x - 5y + 2 = 0$ is a pair of straight lines

24.a. If sin A = $\frac{1}{\sqrt{10}}$ and sin B $\frac{1}{\sqrt{5}}$ prove that $A+B=\frac{\pi}{4}$

b. If A+B = 45° Prove that (1 + tanA) (1 + tanB) = 2 and hence deduce the value of tan $22\frac{1}{2}$

c. Prove that $\frac{\sin 2A+\sin A}{1+\cos 2A+\cos A}=\tan A$

25.a. Prove that sin20°sin40°sin60°sin80° $\frac{3}{16}$

b. Prove that $(\cos\alpha-\cos\theta\beta)^2(\sin\alpha - \sin\beta)^2= 4\sin^2\left\{\frac{\alpha-\beta}{2}\right\}$

c. Show that $\tan^{-1}\left\{\frac{x-y}{1+xy}\right\}=\tan^{-1}x+\tan^{-1}y$

MODEL QUESTION PAPER – 2

Time : 3 Hrs (Maximum Marks: 75)

PART – A

(Marks: 15 x 1 = 15)

I. Answer Any 15 Questions

1. Find x if $\begin{vmatrix} x-2 & 0 \\ 0 & x-2 \end{vmatrix} = 0$

2. Prove that $\begin{vmatrix} x & 2x & 3x \\ 4x & 5x & 6x \\ 7x & 8x & 9x \end{vmatrix} = 0$

3. If A = $\begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}$ Find A^2

4. Find the inverse of $\begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

5. Find the 11th term of $\left(3x^2 + \frac{1}{x}\right)^{20}$

6. How many middle terms are in the expansion of $(5x-y)^9$

7. Write the first three terms in the expansion of $(1-x)^{-2}$

8. Without finding the constants split $\frac{x^3 - x}{(x+2)(x^2+1)}$ in to partial fraction

9. Find the perpendicular distances from the point (2,1) to the straight line 3x+2y+1=0

10. Write down the condition for the pair of lines given by $ax^2+2hxy+by^2 = 0$ to be parallel

11. Find 'a' if the lines represented by$3x^2+4xy+ay^2 = 0$ are perpendicular

12. State the expression for angle between pair of line given by $ax^2+2hxy+by^2 +2gx +2fy+c=0$

13. Find the value sin 15° without using tables or calculator
14. Find the value of cos40°cos 10° + sin 40° sin 10°
15. Simplify $\frac{\tan 22+\tan 23}{1-\tan 22 \tan 23}$
16. Prove that $\frac{\sin 2A}{1+\cos 2A}=\tan A$
17. Find the value of 3sin10° - 4 $\sin^3$10°
18. If Cos A = $\frac{3}{5}$ find the value of cos3A.
19. Show that Cos20° + Cos100° + Cos140° = 0
20. Show that $\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x$

PART- B

(Answer any two subdivision in each Question)

All Questions carry Equal Marks

5x12=60

21 a) Solve the equations 4x+y+z=6,2x-y-2z = -6 and x+y+z=3, using Cramer's rule

b) Prove that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}=x^2(x+a+b+c)$

c) If A = $\begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}$ show that $A^2 - 5A - 14I = 0$

22.a) Find the middle terms in the expansion of $(2x+1/x)^{13}$

b) Find the term independent of 'x' in the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$

c) Resolve $\frac{7x-4}{(x+2)(x-1)}$ in to a partial function

23.a) Derive the expression for angle between two lines $y=m_1x+c_1$ and $y = m_2x+c_2$

b) If the slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other show that $8h^2 = 9ab$

c) Find the value of 'λ' so that the equation $3x^2+14xy+8y^2-8x-2y+\lambda = 0$ represents a pair ofstraight lines.

24.a) If $\sin A = \frac{3}{5}, \cos B=\frac{12}{13},$ find the values of sin (A-B) and cos (A-B)

b) If A + B = 45° Prove that (cot A-1) (cot B -1)=2. Also find the value of cot $22\frac{1}{2}$

c) Show that $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\theta/2$

25.a) Prove that $\frac{\cos^3 A-\cos^3 A}{\cos A} + \frac{\sin^3 A+\sin 3A}{\sin A}=3$

b) If a=sinA + sinB, b = cosA + cosB, Show that

$$\tan^2\frac{A-B}{2} = \frac{4-(a^2+b^2)}{a^2+b^2}$$

c) Show that $\tan^{-1}\frac{3x-x^3}{1-3x^2} = 3\tan^{-1}x$

MATHEMATICS – II

UNIT – I

CIRCLES

1.1 Equation of circle – given centre and radius. General Equation of circle – finding center and radius. Simple problems.

1.2 Equation of circle through three non collinear points – concyclic points. Equation of circle on the line joining the points (x_1,y_1) and (x_2,y_2) as diameter. Simple problems.

1.3 Length of the tangent. Position of a point with respect to a circle. Equation of tangent (Derivation not required). Simple problems.

1.1 CIRCLES

Definition:

The locus or path of a point P(x,y) which is at a constant distance 'r' from a fixed point C(h,k) is called a circle.

The fixed point C(h,k) is called centre and the constant distance is called the radius of the circle.

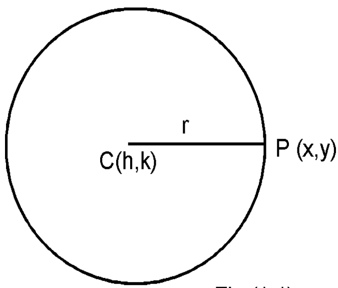


Fig (1.1)

1.1.1 Eqution of a circle with centre (h,k) and radius r:

Let the given centre and radius are C(h,k) and 'r' units. Let P(x,y) be any point on the circle. From Fig (1.1.) CP= r

ie $\sqrt{(x-h)^2+(y-k)^2}=r$ (using distance formula)

$$(x-h)^2+(y-k)^2=r^2 \qquad (1)$$

Note: When centre is at the origin (0,0) the equation (1) becomes $x^2+y^2=r^2$. i.e. the equation of the circle with centre at the origin and radius 'r' units is $x^2+y^2=r^2$.

1.1.2 General Equation of the circle.

The general equation of the circle is

$$x^2+y^2+2gx+2fy+c=0 \qquad (2)$$

Equation (2) can be re written as

$$x^2+2gx+g^2+y^2+2fy+f^2=g^2+f^2-c \text{ (or)}$$

$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

$$[x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2 \qquad (3)$$

Equation (3) is in the form of equation (1)

$\therefore$ The eqution (2) represents a circle with centre (-g,-f) and radius $\sqrt{g^2+f^2-c}$.

Note:

(i) Coefficient of x^2 = coefficient of y^2

(ii) Centre of the circle = $\left[-\frac{1}{2}\text{coefficient of } x, -\frac{1}{2}\text{coefficient of } y\right]$

(iii) Radius = $\sqrt{g^2+f^2-c}$

WORKED EXAMPLES

PART - A

1) Find the equation of the circle whose centre is (2,-1) and radius 3 units.

Solution:

Equation of the circle with centre (h,k) and radius 'r' is

$(x-h)^2+(y-k)^2=r^2$ (h,k) =(2,-1) r=3

$\therefore (x-2)^2+(y+1)^2=3^2$ $x^2+y^2-4x+2y-4=0$

2) Find the centre and radius of the circle $x^2+y^2-6x+4y+2=0$

Solution:

Here 2g = -6 and 2f = 4

g = -3 f = 2

Centre is (-g,-f)

Centre (3,-2)

$\therefore$ r = $\sqrt{(-3)^2+2^2-2}$

$\therefore$ r = $\sqrt{11}$ units

PART – B

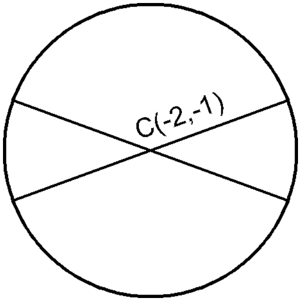
1) If 3x-y+5 = 0 and 4x+7y+15=0 are the equations of two diameters of a circle of radius 4 units write down the equation of the circle.

Solution:

Given diameters are

3x-y+5=0 (1)

4x+7y+15 = 0 (2)



Solving (1) and (2) we get x=-2 and y = -1

$\therefore$ centre (-2,-1) given radius r = 4

$\therefore$ Equation of the circle is

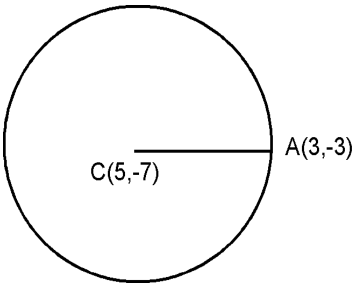
$(x-h)^2+(y-k)^2=r^2$

$(x+2)^2+(y+1)^2=4^2$

i.e. $x^2+y^2+4x+2y-11=0$

2) Find the equation of the circle whose centre is (5,-7) and passing through the point (3,-3)

Solution:



Let the centre and a point on the circle be C(5,-7) and A (3,-3)

$\therefore \text{Radius} = CA = \sqrt{(5-3)^2+(-7+3)^2}$

$r=\sqrt{4+16}$ $\qquad$ $r=\sqrt{20}$

Equation of circle is $(x-h)^2+(y-k)^2=r^2$

$(h,k) = (5,-7)$ $\quad r=\sqrt{20} \quad r^2 = 20$

$\therefore (x-5)^2+(y+7)^2=20$

$x^2-10x+25+y^2+14y+49-20=0$

i.e. $x^2+y^2-10x+14y+54=0$

1.2 CONCYCLIC POINTS

If four or more points lie on the same circle the points are called concylic points.

1.2.1 Equation of circle with end points of a diameter

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be given two end points of a diameter. Let P (x,y) be any point on the circle.

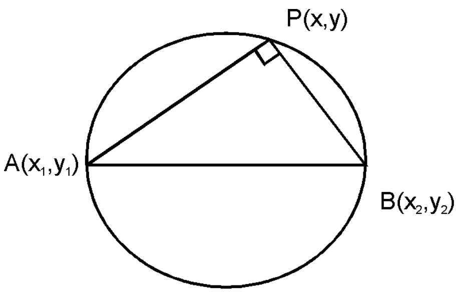


Fig (1.2)

i.e $\angle APB=90^\circ$ ($\because$ Angle in a Semi – circle is 90°)

$AP \perp PB$

$\therefore$ (slope of AP) (slope of PB) = -1

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right)=-1$$

ie. $(y-y_1)(y-y_2)=-(x-x_1)(x-x_2)$

ie $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

is the required equation of the circle.

WORKED EXAMPLES

PART – A

1) Find the equation of the circle joining the points (1,-1) and (-2,3) as diameter

Solution:

Equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

$$(x_1 y_1)=(1,-1) \qquad (x_2,y_2)=(-2,3)$$

$$\therefore (x-1)(x+2)+(y+1)(y-3)=0$$

$$x^2+y^2+x-2y-5=0$$

2) Find the equation of the circle joining the points (a,0) and (0,b) as diameter

Solution:

Equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

$$(x_1,y_1)=(a,0)\,\text{and}\,(x_2,y_2)=(0,b)$$

$$\therefore (x-a)(x-0)+(y-0)(y-b)=0$$

$$x^2+y^2-ax-by=0$$

PART – B

1) Find the equation of the circle passing through the points (1,1), (1,0) and (0,1)

Solution:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

(1,1) lies on (1) i.e

$$1^2 + 1^2 + 2g(1) + 2f(1) + c = 0$$

$$2g+2f+c = -2 \qquad (2)$$

(1,0) lies on (1)

i.e $1^2 + 0^2 + 2g(1) + 2f(0) + c = 0$

$$2g+c = -1 \qquad (3)$$

(0,1) lies on (1)

i.e $0^2 + 1^2 + 2g(0) + 2f(1) + c = 0$

$$2f+c = -1 \qquad (4)$$

$$\begin{array}{rl} & 2g+2f+c = -2 \\ & 2g+0+c=-1 \\ \hline (2) - (3) \Rightarrow & 0 + 2f+0=-1 \\ \hline & 2f=-1 \quad \therefore f= -\frac{1}{2} \end{array}$$

Substitute $f=-\frac{1}{2}$ in (4)

$$2\left(-\frac{1}{2}\right) + c = -1$$

$$-1 + c = -1 \;\therefore c = 0$$

Substitute c= 0 in (3)

$$2g+0=-1$$

$$g=-\frac{1}{2}$$

$\therefore$ Equation of the circle is

$$x^2+y^2+2\left(\frac{-1}{2}\right)x+2\left(\frac{-1}{2}\right)y+0=0$$

ie. $x^2+y^2-x-y=0$

2) Find the equation of the circle passing through the points (0,1) , (4,3) and having its centre on the line 4x-5y-5=0

Solution:

Let the Equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad (1)$$

(0,1) lies on (1)

i.e $0^2+1^2+2g(0)+2f(1)+c=0$

$$2f+c = -1 \qquad (2)$$

(4,3) lies on (1)

i.e. $4^2+3^2+2g(4)+2f(3)+c=0$

$$8g+6f+c=-25 \qquad (3)$$

Centre (-g,-f) lies on the line $4x-5y-5=0$

i.e $4(-g)-5(-f)-5=0 \qquad (4)$

$$-4g+5f=5$$

$$2f+c=-1$$

$$8g+6f+c=-25$$

(2) – (3)$\Rightarrow$ $\quad \overline{\underline{-8g - 4f + 0 = 24}}$

$$4g + 2f = -12 \qquad (5)$$

(4) + (5) gives;

$$\begin{aligned} -4g + 5f &= 5 \\ 4g + 2f &= -12 \\ \hline 0 + 7f &= -7 \\ 7f &= -7 \end{aligned} \qquad \therefore f = -1$$

Substituting f=-1 in (5)

4g-2 (-1) = -12

4g+2 = -12 4g = -10 $\therefore g = \dfrac{-10}{4}$ i.e., $g = \dfrac{-5}{2}$

Substituting $g = \dfrac{-5}{2}$, f=-1 in (3)

$$8\left(\frac{-5}{2}\right) + 6(-1) + c = -25$$

$-20 - 6 + c = -25$ $\quad c = -25 + 26$ $\qquad$ c = 1

$\therefore$ Equation of the required circle is

$$x^2 + y^2 + 2\left(\frac{-5}{2}\right)x + 2(-1)y + 1 = 0$$

$$x^2 + y^2 - 5x - 2y + 1 = 0$$

3) Show that the points (4,1), (6,5) (2,7) and (0,3) are concyclic

Solution:

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

(4,1) lies on the circle (1)

$$4^2 + 1^2 + 2g(4) + 2f(1) + c = 0$$

$$8g + 2f + c = -17 \qquad (2)$$

(6,5) lies on the circle (1)

$$\therefore\ 6^2+5^2+2g(6)+2f(5)+c=0$$

$12g+10f+c = -61$ (3)

(2,7) lies on the circle (1)

$$\therefore\ 2^2+7^2+2g(2)+2f(7)+c=0$$

$4g+14f+c = -53$ (4)

$(3) - (2) \Rightarrow\ 4g + 8f = -44$ (5)

$(3)-(4) \Rightarrow 8g - 4f = -8$ (6)

Solving (5) and (6) we get $g=-3$ and $f=-4$

Substituting $g = -3$, $f=-4$ in (2)

$8(-3) + 2(-4)+c = -17$

$C = -17 +32 = 15$

$\therefore$ Equation of circle passing through the three points (4,1) (6,5) and (2,7) is

$$x^2+y^2+2(-3)x+2(-4)y+15=0$$

$$x^2+y^2-6x-8y+15=0 \qquad (7)$$

Substituting the fourth point (0,3) in (7)

$$0^2+3^2-6(0)-8(3)+15=0$$

$$9-24+15=0 \qquad \therefore\ 24 -24 = 0$$

$\therefore$ (0,3) also lies on (7)

Hence the given four points are concyclic.

1.3.1 Length of the Tangent to a circle from a point (x_1,y_1)

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$ and the point $A(x_1,y_1)$ lies out side the circle.

We know that the centre is C(-g,-f) and radius $BC = r = \sqrt{g^2 + f^2 - c}$

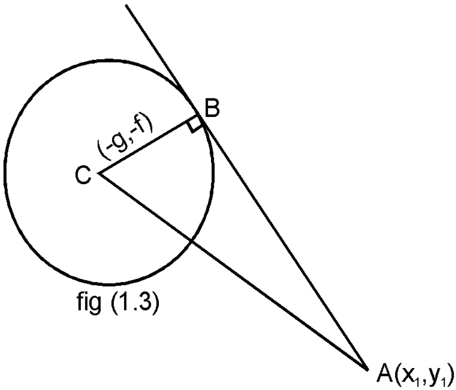


fig (1.3)

From the fig (1.3)

Δ ABC is a right angled triangle.

$\therefore\ AB^2 + BC^2 = AC^2$

$AB^2 = AC^2 - BC^2$

$AB^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$

$AB^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + C$

$AB = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Which is the length of the tangent from the point (x_1,y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Note:

(i) If AB > 0, $A(x_1,y_1)$ lies out side the circle.

(ii) If AB < 0, $A(x_1,y_1)$ lies inside the circle.

(iii) If AB = 0, A (x_1,y_1) lies on the circle.

1.3.2 Equation of the Tangent to a circle at the point (x_1,y_1) on the circle (Results only):

Given equation of circle is $x^2+y^2+2gx+2fy+c=0$

The point A (x_1,y_1) lies on the circle.

i.e $x_1^2+y_1^2+2gx_1+2fy_1+c=0$

From fig (1.4) AT is the tangent at A. We know that centre is C(-g,-f).

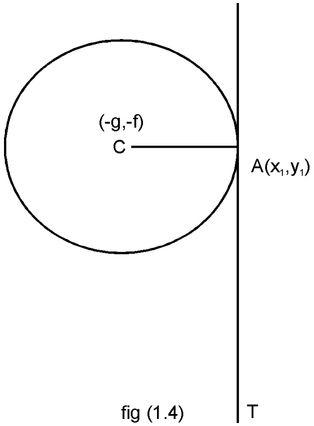


fig (1.4)

Slope of AC = $\dfrac{y_1+f}{x_1+g}$

Since AC is perpendicular to AT

Slope of AT = m = $-\dfrac{(x_1+g)}{(y_1+f)}$

$\therefore$ Equation of the tangent AT at A(x_1,y_1) is

$$y-y_1=m(x-x_1)$$

$$y-y_1=-\left(\frac{x_1+g}{y_1+f}\right)(x-x_1)$$ on simplification, we get

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

Note: The equation of tangent to the circle $x^2+y^2=r^2$ at (x_1,y_1) is obtained by substituting g=0, f=0 and c=-r^2 in the above equation to tangent

$\therefore\ xx_1 + yy_1 + 0(x + x_1) + 0(y + y_1) - r^2 = 0$

ie $xx_1 + yy_1 = r^2$

1.3.2 RESULTS

1) Equation of the tangent to a circle at a point (x_1,y_1) is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

2) Length of the tangent from the point (x_1,y_1) to the circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Note:

(1) the equation of the tangent to the circle $x^2+y^2 = r^2$ at (x_1,y_1) is $xx_1 + yy_1 = r^2$

(2) If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$ then the point (x_1,y_1) lies outside the circle.

(3) If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ then the point (x_1,y_1) lies on the circle.

(4) If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$, then the point (x_1,y_1) lies inside the circle.

WORKED EXAMPLES

PART – A

1) Find the length of the tangent from (2,3) to the circle
$x^2 + y^2 - 2x + 4y + 1 = 0$

Solution:

Length of the tangent = $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$= \sqrt{2^2 + 3^2 - 2(2) + 4(3) + 1}$

$= \sqrt{4 + 9 - 4 + 12 + 1}$

$= \sqrt{22}$ units

2) Show that the point (9,2) lies on the circle
$x^2+y^2-6x-10y-11=0$

Solution:

Substitute (9,2) on the circle $x^2+y^2-6x-10y-11=0$

$9^2+2^2-6(9)-10(2)-11=0$

81 + 4-54-20-11=0. $\therefore$ 85-85 = 0

$\therefore$ the point (9,2) lies on the circle.

3) Find the equation of the tangent at (-4,3) to the circle $x^2+y^2=25$

Solution:

Equation of tangent is $xx_1+yy_1=r^2$

$\therefore$ x(-4) + y(3) = 25

-4x+3y =25

4x-3y+25 = 0

PART – B

1) Find the equation of the tangent at (4,1) to the circle
$x^2+y^2-8x-6y+21=0$

Solution:

Equation of the tangent at the point (x_1,y_1) is
$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

Given $x^2+y^2-8x-6y+21=0$

2g = -8, 2f = -6 c = 21 (x_1,y_1) = (4,1)
g = -4 f =-3

$x(4)+y(1)+(-4)[x+4]+(-3)[y+1]+(21)=0$

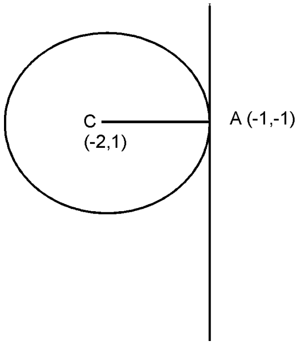
$4(x)+y-4x-16-3y-3+21=0$

-2y+2 = 0 2y-2 = 0 $\therefore$y-1 = 0

Equation of the tangent is y -1 = 0

2) Find the equation of the tangent to the circle $(x+2)^2 + (y-1)^2 = 5$ at (-1,-1).

Solution:



Equation of the circle in $(x+2)^2 + (y-1)^2 = 5$

i.e $x^2 + y^2 + 4x - 2y = 0$

Here 2g=4 2f = -2

g=2 f=-1 c= 0

$\therefore$ Equation of the tangent is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

i.e., $x(-1) + y(-1) + 2(x-1) - 1(y-1) = 0$

$-x - y + 2x - 2 - y + 1 = 0$ x-2y-1 = 0

EXERCISE
PART – A

1. Find the equation of the circle whose centre and radius are given as

(i) (3,2); 4 units (ii) (-5,7) , 3 units

(iii) (-5,-4); 5 units (iv) (6,-2), 10 units

2. Find the centre and radius of the following circles:

(i) $x^2+y^2-12x-8y+2=0$ (ii) $x^2+y^2+7x+5y-1=0$

(iii) $2x^2+2y^2-6x+12y-4=0$ (iv) $x^2+y^2=100$

3. Write down the equation of circle whose centre is (h,k) and radius 'r' units.

4. Write down the centre and radius of the circle $x^2+y^2+2gx+2fy+c=0$

5. Find the centre and radius of the circle $(x-2)^2+(y+3)^2=16$

6. Find the equation of the circle described on line joining the following points as diameter:

(i) (3,5) and (2,7) (ii) (-1,0) and (0,-3)

(iii) (0,0) and (4,4) (iv) (-6,-2) and (-4,-8)

7. Write down the equation of the circle whose end points of the diameter are (x_1,y_1) and (x_2,y_2)

8. Write down the expression to find the length of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ from the point (x_1,y_1)

9. Write down the equation of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ at the point (x_1,y_1).

10. Find the length of the tangent from the point (2,1) to the circle $x^2+y^2+2x+4y+3=0$

11. Show that the point (-3,-4) lies inside the circle $x^2+y^2+2x+y-25=0$

12. Show that the point (-1,-7) lies on the circle $x^2+y^2+15x+2y-21=0$

PART – B

1) x+2y=1 and 3x-4y=3 are two diameters of a circle of radius 5 units. Find the equation of the circle.

2) Find the equation of the circle two of its diameters are 3x+4y=2 and x-y =3 and passing through (5,-1)

3) Find the equation of the circle passing through the points (5,2), (2,1), (1,4).

4) Find the equation of the circle passing through the points (6,0) and (-1,-1) and having its centre on x+2y+5=0

5) Prove that the points (3,4), (0,5),(-3,-4) and (-5,0) are concylic

6) Find the equation of the tangent at (2,4) to the circle $x^2 + y^2 + 2x - 4y - 8 = 0$

7) Find the equation of the tangent at (-7,-11) to the circle $x^2 + y^2 = 500$

8) Show that the point (1,-4) lies on the circle $x^2 + y^2 - 12x + 4y + 11 = 0$ Also find the equation of the tangent at (1,-4).

ANSWER

PART – A

(1) (i) $x^2 + y^2 - 6x - 4y - 3 = 0$

(ii) $x^2 + y^2 + 10x - 14y + 65 = 0$

(iii) $x^2 + y^2 + 10x + 8y + 16 = 0$

(iv) $x^2 + y^2 - 12x + 4y - 60 = 0$

(2) (i) (6,4), $\sqrt{50}$ (ii) $\left(\frac{-7}{2},\frac{-5}{2}\right)$; $\frac{\sqrt{78}}{2}$

(iii) $\left(\frac{3}{2},-3\right)$; $\frac{\sqrt{53}}{2}$ (iv) (0,0); 10.

(5) (2,-3), 4

(6) (i) $x^2+y^2-5x-12y+41=0$

(ii) $x^2+y^2+x+3y=0$

(iii) $x^2+y^2-4x-4y=0$

(iv) $x^2+y^2+10x+10y+40=0$

(10) 4 units

(11) $\sqrt{-10}$

PART – B

1) $x^2+y^2-2x-24=0$

2) $x^2+y^2-4x+2y-4=0$

3) $x^2+y^2-6x-6y+13=0$

4) $x^2+y^2-6x+8y=0$

6) $3x+2y-14=0$

7) $7x+11y+170=0$

8) $5x+2y+3=0$

UNIT- II

FAMILY OF CIRCLES

2.1 Concentric circles – contact of circles (internal and external circles) – orthogonal circles – condition for orthogonal circles. (Result only). Simple Problems

2.2 **Limits:**Definition of limits - $\underset{x \to a}{\text{Lt}} \dfrac{x^n - a^n}{x-a} = na^{n-1}$

$\underset{\theta \to 0}{\text{Lt}} \dfrac{\sin\theta}{\theta} = 1,$ $\quad \underset{\theta \to 0}{\text{Lt}} \dfrac{\tan\theta}{\theta} = 1$ $\quad (\theta \text{ in radian})$

[Results only] – Problems using the above results.

Differentiation:

2.2 Definition – Differentiation of x^n, sinx, cosx, tanx, cotx, secx, cosecx, logx, e^x, $u \pm v$, uv, uvw, $\dfrac{u}{v}$ (Results only). Simple problems using the above results.

2.1. FAMILY OF CIRCLES

2.1.1 Concentric Circles.

Two or more circles having the same centre are called concentric circles.

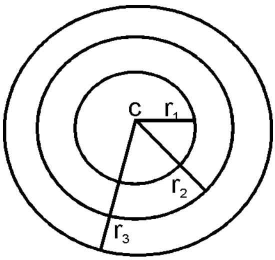


Fig (2.1)

Equation of the concentric circle with the given circle

$x^2+y^2+2gx+2fy+c = 0$ is $x^2+y^2+2gx+2fy+k = 0$

(Equation differ only by the constant term)

2.1.2 Contact of Circles.

Case (i) Two circles touch externally if the distance between their centers is equal to sum of their radii.

i.e. $c_1c_2 = r_1 + r_2$

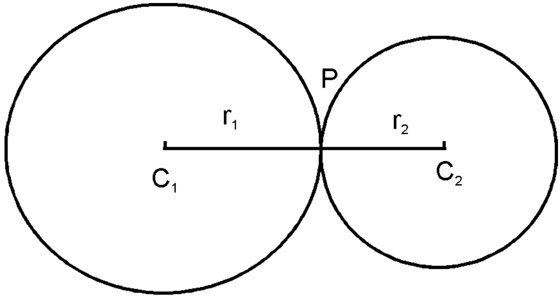


Fig (2.2)

Case (ii) Two circles touch internally if the distance between their centers is equal to difference of their radii.

i.e. $c_1c_2 = r_1 - r_2$ (or) $r_2 - r_1$

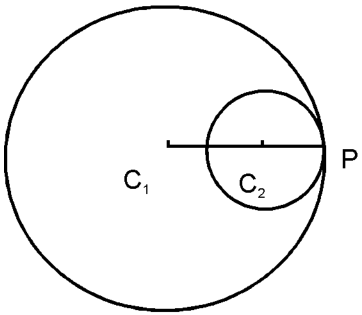


Fig (2.3)

Orthogonal Circles

Two circles are said to be orthogonal if the tangents at their point of intersection are perpendicular to each other.

2.1.3 Condition for Two circles to cut orthogonally.(Results only)

Let the equation of the two circles be

$$X^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$X^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

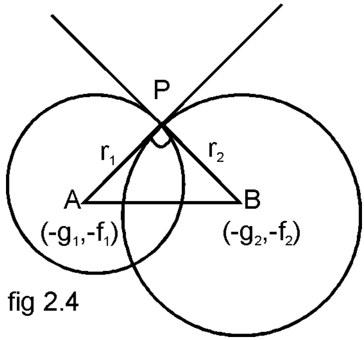


fig 2.4

They cut each other orthogonally at the point P.

The centers and radii of the circles are

$A\ (-g_{1,} - f_1)$, $B\ (-g_{2,} - f_2)$

$AP = r_1 = \sqrt{g_1{}^2 + f_1{}^2 - c_1}$ and $BP = r_2 = \sqrt{g_2{}^2 + f_2{}^2 - c_2}$

From fig (2.4) Δ APB is a right angled triangle,

$$AB^2 = AP^2 + PB^2$$

i.e. $(-g_1 + g_2)^2 + (-f_1 + f_2)^2 = g_1{}^2 + f_1{}^2 - c_1 + g_2{}^2 + f_2{}^2 - c_2$

Expanding and simplifying we get,

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$ is the required condition for two circles to cut orthogonally.

Note: When the center of any one circle is at the origin then condition for orthogonal circles is $c_1+c_2=0$

2.1 WORKED EXAMPLES

PART – A

1. Find the distance between the centre of the circles

   $x^2 + y^2 – 4x + 6y + 8 = 0$ and $x^2 + y^2 – 10x - 6y + 14 = 0$

Solution:

$x^2 + y^2 – 4x + 6y + 8 = 0$ and $x^2 + y^2 – 10x - 6y + 14 = 0$

centre: $c_1 (2, -3)$ $c_2 (5,3)$

$\therefore$ Distance= $c_1c_2 = \sqrt{(2-5)^2 + (-3-3)^2}$

$c_1c_2 = \sqrt{45}$

2. Find the equation of the circle concentric with the circle

   $x^2+y^2–25 = 0$ and passing through (3,0).

Solution:

Equation of concentric circle with

$x^2 + y^2 – 25 = 0$ is

$x^2 + y^2 + k = 0$ which passes through (3,0)

i.e. $3^2 + 0^2 + k = 0$

$k = -9$

$\therefore$ Required Equation of the circle is

$x^2 + y^2 – 9 = 0$

3. Find whether the circles $x^2 + y^2 + 15 = 0$ and

   $x^2 + y^2 – 25 = 0$ cut orthogonally or not.

Solution:

When any one circle has centre at origin, orthogonal condition is

$c_1 + c_2 = 0$

i.e. $15 - 25 \neq 0$

Given circles do not cut orthogonally.

PART - B

1. Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x + 8y + 4 = 0$ and having radius 3 units.

Solution:

Centre of the circle $x^2 + y^2 - 4x + 8y + 4 = 0$ is (2,-4).

$\therefore$Centre of concentric circle is (2,-4) and radius r = 3. Equation of the required circle is

$(x - h)^2 + (y - k)^2 = r^2$

$(x - 2)^2 + (y + 4)^2 = 3^2$

$x^2 - 4x + 4 + y^2 + 8y + 16 = 9$

$x^2 + y^2 - 4x + 8y + 11 = 0$

2. Show that the circles $x^2 + y^2 - 4x - 6y + 9 = 0$ and $x^2 + y^2 + 2x + 2y - 7 = 0$ touch each other.

Solution:

Given circles

$x^2 + y^2 - 4x - 6y + 9 = 0$ and $x^2 + y^2 + 2x + 2y - 7 = 0$

centre: c_1 (2,3) $\qquad$ c_2(-1,-1)

radius:

$r_1 = \sqrt{2^2 + 3^2 - 9}$ $\qquad$ $r_2 = \sqrt{(+1)^2 + 1^2 + 7}$

$= \sqrt{4}$ $\qquad$ $r_2 = \sqrt{9}$

$r_1 = 2$ $\qquad$ $r_2 = 3$

Distance:

$$c_1c_2 = \sqrt{(2+1)^2 + (3+1)^2}$$

$$= \sqrt{3^2 + 4^2}$$

$$= \sqrt{25}$$

$c_1c_2 = 5$

$c_1c_2 = r_1 + r_2$

$\therefore$ The circles touch each other externally.

3. Find the equation of the circle which passes through (1,1) and cuts orthogonally each of the circles

$x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$.

Solution:

Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ 1

This passes through (1,1)

i.e. $1^2+1^2 + 2g(1) + 2f(1) + c = 0$

$2g + 2f + c = -2$ 2

Equation (1) orthogonal with the circle

$x^2 + y^2 - 8x - 2y + 16 = 0$

$2g_1=2g$	$2f_1=2f$	$c_1=c$
$g_1=g$	$f_1=f$	
$2g_2=-8$	$2f_2=-2$	$c_2=16$
$g_2=-4$	$f_2=-1$	

by orthogonal condition

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

i.e. $2g(-4) + 2f(-1) = c + 16$

$-8g - 2f - c = 16$

$8g + 2f + c = -16$ 3

Equation (1) orthogonal with the circle

$x^2 + y^2 - 4x - 4y - 1 = 0$

i.e. $2g(-2) + 2f(-2) = c - 1$

$-4g - 4f - c = -1$

$4g + 4f + c = 1$ 4

(3) - (2)

$$\begin{array}{l} 8g + 2f + c = -16 \\ \underline{2g + 2f + c = -2} \\ 6g \qquad\quad = -14 \end{array}$$

$$g = \frac{-14}{6}$$

$$g = \frac{-7}{3}$$

(4) - (2)

$$4g + 4f + c = 1$$
$$\underline{2g + 2f + c = -2}$$
$$2g + 2f \quad = 3 \qquad 5$$

i.e. put $g = \frac{-7}{3}$ in (5)

$$2\left(\frac{-7}{3}\right) + 2f = 3$$
$$2f = 3 + \frac{14}{3}$$
$$f = \frac{23}{6}$$

Substitute $g = \frac{-7}{3}$ and $f = \frac{23}{6}$ in (1)

$$2\left(\frac{-7}{3}\right) + 2\left(\frac{23}{6}\right) + c = -2$$

Simplifying we get $c = \frac{-15}{3}$

$\therefore$ Required equation of circle is

$$x^2 + y^2 + 2\left(\frac{-7}{3}\right)x + 2\left(\frac{23}{6}\right)y - \frac{15}{3} = 0$$
$$x^2 + y^2 - \frac{14}{3}x + \frac{23}{3}y - \frac{15}{3} = 0$$

or

$$3x^2 + 3y^2 - 14x + 23y - 15 = 0$$

2.2 LIMITS

Introduction

The concept of function is one of the most important tool in calculus. Before, we need the following definitions to study calculus.

Constant:

A quantity which retains the same value throughout a mathematical process is called a constant, generally denoted by a,b,c,…

Variable:

A quantity which can have different values in a particular mathematical process is called a variable, generally denoted by x,y,z,u,v,w.

Function:

A function is a special type of relation between the elements of one set A to those of another set B Symbolically f: A $\to$ B

To denote the function we use the letters f,g,h…. Thus for a function each element of A is associated with exactly one element in B. The set A is called the domain of the function f and B is called co-domain of the function f.

2.2.1. Limit of a function.

Consider the function f: A $\to$ B is given by

$f(x) = \frac{x^2 - 1}{x - 1}$ when we put x = 1

We get $f(x) = \frac{0}{0}$ (Indeterminate form)

But constructing a table of values of x and f(x) we get

X	0.95	0.99	1.001	1.05	1.1	1.2
F(x)	1.95	1.99	2.001	2.05	2.1	2.2

From the above table, we can see that as 'x' approaches (nearer) to 1, f(x) approaches to 2.

It is denoted by $\underset{x \to 1}{\text{Lt}} \frac{x^2 - 1}{x - 1} = 2$

We call this value 2 as limiting value of the function.

2.2.2 Fundamental results on limits.

1) $\underset{x \to a}{\text{Lt}} [f(x) \pm g(x)] = \underset{x \to a}{\text{Lt}} f(x) \pm \underset{x \to a}{\text{Lt}} g(x)$

2) $\underset{x \to a}{\text{Lt}} [f(x)g(x)] = \underset{x \to a}{\text{Lt}} f(x) \underset{x \to a}{\text{Lt}} g(x)$

3) $$\underset{x \to a}{Lt} \frac{f(x)}{g(x)} = \frac{\underset{x \to a}{Lt} f(x)}{\underset{x \to a}{Lt} g(x)}$$

4) $$\underset{x \to a}{Lt} Kf(x) = K \underset{x \to a}{Lt} f(x)$$

5) If $f(x) \le g(x)$ then

$$\underset{x \to a}{Lt} f(x) \le \underset{x \to a}{Lt} g(x)$$

Some Standard Limits.

1) $$\underset{x \to a}{Lt} \frac{x^n - a^n}{x - a} = na^{n-1} \quad \text{(for all values of } n\text{)}$$

When 'θ' is in radian 2) $$\underset{\theta \to 0}{Lt} \frac{\sin\theta}{\theta} = 1$$

Note: (1) $$\underset{\theta \to 0}{Lt} \frac{\tan\theta}{\theta} = 1$$ (2) $$\underset{\theta \to 0}{Lt} \frac{\sin n\theta}{\theta} = n$$

(3) $$\underset{\theta \to 0}{Lt} \frac{\tan n\theta}{\theta} = n$$

2.2 WORKED EXAMPLES
PART – A

1. Evaluate:

$$\underset{x \to 0}{Lt} \frac{3x^2 + 2x + 1}{5x^2 + 6x + 7}$$

Solution:

$$\underset{x \to 0}{Lt} \frac{3x^2 + 2x + 1}{5x^2 + 6x + 7} = \frac{3(0)^2 + 2(0) + 1}{5(0)^2 + 6(0) + 7} = \frac{1}{7}$$

2. Evaluate:

$$\underset{x \to 1}{Lt} \frac{x^2 + 4x - 5}{x^2 + x - 2}$$

Solution:

$$\underset{x\to 1}{\text{Lt}}\frac{x^2+4x-5}{x^2+x-2}=\underset{x\to 1}{\text{Lt}}\frac{(x+5)(x-1)}{(x+2)(x-1)}$$

$$=\underset{x\to 1}{\text{Lt}}\frac{(x+5)}{(x+2)}=\underset{x\to 1}{\text{Lt}}\frac{x+5}{x+2}=\frac{6}{3}=2$$

3. Evaluate:

$$\underset{x\to 2}{\text{Lt}}\frac{x^n-2^n}{x-2}$$

Solution:

$$\underset{x\to 2}{\text{Lt}}\frac{x^n-2^n}{x-2}=n2^{n-1} \qquad \because \underset{x\to a}{\text{Lt}}\frac{x^n-a^n}{x-a}=na^{n-1}$$

4. $$\underset{x\to a}{\text{Lt}}\frac{\sqrt{x}-\sqrt{a}}{x-a}$$

Solution:

$$\underset{x\to a}{\text{Lt}}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\underset{x\to a}{\text{Lt}}\frac{x^{\frac{1}{2}}-a^{\frac{1}{2}}}{x-a}$$

$$=\frac{1}{2}a^{\frac{1}{2}-1}=\frac{1}{2}a^{-\frac{1}{2}}$$

5. Evaluate:

$$\underset{x\to 0}{\text{Lt}}\frac{\sin 5x}{3x}$$

Solution:

$$\underset{x\to 0}{\text{Lt}}\frac{\sin 5x}{3x}=\frac{1}{3}\underset{x\to 0}{\text{Lt}}\frac{5\sin 5x}{5x}=\frac{5}{3}\underset{x\to 0}{\text{Lt}}\frac{\sin 5x}{5x}=\frac{5}{3} \qquad \because \underset{\theta\to 0}{\text{Lt}}\frac{\sin\theta}{\theta}=1$$

PART - B

1. Evaluate:

$$\underset{x \to 4}{\text{Lt}} \frac{x^3 - 64}{x^2 - 16}$$

Solution:

$$\underset{x \to 4}{\text{Lt}} \frac{x^3 - 64}{x^2 - 16} = \underset{x \to 4}{\text{Lt}} \frac{x^3 - 4^3}{x^2 - 4^2} = \underset{x \to 4}{\text{Lt}} \frac{\frac{x^3 - 4^3}{x - 4}}{\frac{x^2 - 4^2}{x - 4}}$$

$$= \frac{\underset{x \to 4}{\text{Lt}} \frac{x^3 - 4^3}{x - 4}}{\underset{x \to 4}{\text{Lt}} \frac{x^2 - 4^2}{x - 4}} = \frac{3(4)^2}{2(4)}$$

$$= \frac{48}{8} = 6$$

2. Evaluate:

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin a\theta}{\sin b\theta}$$

Solution:

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin a\theta}{\sin b\theta} = \underset{\theta \to 0}{\text{Lt}} \frac{\frac{\sin a\theta}{\theta}}{\frac{\sin b\theta}{\theta}}$$

$$= \frac{\underset{\theta \to 0}{\text{Lt}} \frac{\sin a\theta}{\theta}}{\underset{\theta \to 0}{\text{Lt}} \frac{\sin b\theta}{\theta}} = \frac{a}{b} \qquad \because \underset{\theta \to 0}{\text{Lt}} \frac{\sin n\theta}{\theta} = n$$

3. Evaluate:

$$\underset{x \to 0}{\text{Lt}} \frac{1 - \cos ax}{1 - \cos bx}$$

Solution:

$$\underset{x \to 0}{\text{Lt}} \frac{1-\cos ax}{1-\cos bx} = \underset{x \to 0}{\text{Lt}} \frac{2\sin^2 a\frac{x}{2}}{2\sin^2 b\frac{x}{2}} \qquad \because 1-\cos 2\theta = 2\sin^2\theta$$

$$= \frac{\underset{x \to 0}{\text{Lt}} \left(\sin\frac{a}{2}x\right)^2}{\underset{x \to 0}{\text{Lt}} \left(\sin\frac{b}{2}x\right)^2}$$

$$= \frac{\underset{x \to 0}{\text{Lt}} \dfrac{\left(\sin\frac{a}{2}x\right)^2}{\left(\frac{a}{2}\right)^2} \times \left(\frac{a}{2}\right)^2}{\underset{x \to 0}{\text{Lt}} \dfrac{\left(\sin\frac{b}{2}x\right)^2}{\left(\frac{b}{2}\right)^2} \times \left(\frac{b}{2}\right)^2} = \frac{\frac{a^2}{4}}{\frac{b^2}{4}} = \frac{a^2}{b^2}$$

2.3. DIFFERENTIATION

Consider a function y = f(x) of a variable x. Let Δx be a small change (positive or negative) in x and Δy be the corresponding change in y.

The differentiation of y with respect to x is defined as limiting value of $\frac{\Delta y}{\Delta x}$, as $\Delta x \to 0$

$$\text{i.e} \quad \frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The following are the differential co-efficient of some simple functions:

1. $\frac{d}{dx}(\text{constant}) = 0$

2. $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

3. $\frac{d}{dx}(x) = 1$

4. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

5. $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$

6. $\frac{d}{dx}(\sin x) = \cos x$

7. $\frac{d}{dx}(\cos x) = -\sin x$

8. $\frac{d}{dx}(\tan x) = \sec^2 x$

9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

10. $\frac{d}{dx}(\sec x) = \sec x \tan x$

11. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

12. $\frac{d}{dx}\left(e^x\right) = e^x$

13. $\frac{d}{dx}(\log x) = \frac{1}{x}$

The following are the methods of differentiation when functions are in addition, multiplication and division.

If u,v and w are functions of x

(i) Addition Rule : $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

$$\frac{d}{dx}(u+v+w) = \frac{d}{dx}(u) + \frac{d}{dx}(v) + \frac{d}{dx}(w)$$

$$\frac{d}{dx}(u-v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$$

(ii) Multiplication Rule : $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

$$\frac{d}{dx}(uvw) = uv\frac{d}{dx}(w) + uw\frac{d}{dx}(v) + vw\frac{d}{dx}(u)$$

(iii) Quotient Rule : (division)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

2.3 WORKED EXAMPLES
PART – A

1. Find $\frac{dy}{dx}$ if $y = \frac{3}{x^2} + \frac{2}{x} + \frac{1}{4}$

Solution:

$$y = 3x^{-2} + 2x^{-1} + \frac{1}{4}$$

$$\therefore \frac{dy}{dx} = -6x^{-3} - 2x^{-2} + 0$$

$$= \frac{-6}{x^3} - \frac{2}{x^2}$$

2. Find $\frac{dy}{dx}$ if $y = e^x \sin x$

Solution: $y = e^x \sin x$

$$\therefore \frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \cos x + \sin x\ e^x$$

3. FInd $\frac{dy}{dx}$ if $y = \frac{x-1}{x+3}$

Solution:

$$y = \frac{x-1}{x+3}$$

$$\therefore \frac{dy}{dx} = \frac{(x+3)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+3)}{(x+3)^2}$$

$$= \frac{(x+3)(1) - (x-1)(1)}{(x+3)^2} = \frac{4}{(x+3)^2}$$

4. Find $\frac{dy}{dx}$ if $y = (x-1)(x-5)(x-3)$

Solution:

$$y = (x-1)(x-5)(x-3)$$

$$\therefore \frac{dy}{dx} = (x-1)(x-5)\frac{d}{dx}(x-3) + (x-1)(x-3)\frac{d}{dx}(x-5)$$
$$+(x-5)(x-3)\frac{d}{dx}(x-1)$$

$$\frac{dy}{dx} = (x-1)(x-5)(1) + (x-1)(x-3)(1) + (x-5)(x-3)(1)$$
$$= (x-1)(x-5) + (x-1)(x-3) + (x-5)(x-3)$$

5. Find $\frac{dy}{dx}$ if $y = x^4 + \frac{1}{\sin x} - \frac{1}{3}$

Solution:

$$y = x^4 + \frac{1}{\sin x} - \frac{1}{3} = x^4 + \operatorname{cosec} x - \frac{1}{3}$$

$$\therefore \frac{dy}{dx} = 4x^3 - \operatorname{cosec} x \cot x$$

PART - B

1. FInd $\frac{dy}{dx}$ if $y = (x^2+3)\cos x \log x$

Solution:

$$y = (x^2+3)\cos x \log x$$

$$\therefore \frac{dy}{dx} = (x^2+3)\cos x\frac{d}{dx}(\log x) + (x^2+3)\log x\frac{d}{dx}(\cos x)$$
$$+\cos x \log x\frac{d}{dx}(x^2+3)$$

$$= (x^2+3)\cos x\left[\frac{1}{x}\right] + (x^2+3)\log x[-\sin x] + \cos x \log x[2x]$$

$$= \frac{(x^2+3)\cos x}{x} - (x^2+3)\log x \sin x + 2x\cos x \log x$$

2. Find $\frac{dy}{dx}$ if $y=\frac{x^3\tan x}{e^x+1}$

Solution:

$$y=\frac{x^3\tan x}{e^x+1}$$

$u=x^3\tan x$ $\quad\quad$ $v=e^x+1$

$\frac{du}{dx}=x^3\left[\sec^2 x\right]+\tan x\left[3x^2\right]$ $\quad\quad$ $\frac{dv}{dx}=e^x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\therefore\frac{dy}{dx}=\frac{\left(e^x+1\right)\left[x^3\sec^2 x+3x^2\tan x\right]-\left(x^3\tan x\right)\left[e^x\right]}{\left(e^x+1\right)^2}$$

3. Find $\frac{dy}{dx}$ if $y=\frac{1+x+x^2}{1-x+x^2}$

Solution:

$$y=\frac{1+x+x^2}{1-x+x^2}$$

$u=1+x+x^2$ $\quad\quad$ $v=1-x+x^2$

$\frac{du}{dx}=0+1+2x$ $\quad\quad$ $\frac{dv}{dx}=0-1+2x$

$$\therefore\frac{dx}{dy}=\frac{\left(1-x+x^2\right)\left[1+2x\right]-\left(1+x+x^2\right)\left[-1+2x\right]}{\left(1-x+x^2\right)^2}$$

$$\frac{dy}{dx}=\frac{2\left[1-x^2\right]}{\left(1-x+x^2\right)^2}$$

4. Find $\frac{dy}{dx}$ if $y=\left(1+\sin x\right)\left(3-\cos x\right)$

Solution:

$$y=\left(1+\sin x\right)\left(3-\cos x\right)$$

$$\therefore \frac{dy}{dx} = (1+\sin x)\frac{d}{dx}(3-\cos x)+(3-\cos x)\frac{d}{dx}(1+\sin x)$$
$$= (1+\sin x)[\sin x]+(3-\cos x)[\cos x]$$

EXERCISE
PART - A

1. Write down the equation of the concentric circle with the circle $x^2+y^2+2gx+2fy+c=0$
2. State the condition for two circles to touch each other externally.
3. State the condition for two circles to touch each other internally.
4. State the condition for two circles to cut each other orthogonally.
5. Find the equation of the circle concentric with the circle $x^2+y^2-2x+5y+1=0$ and passing through the point (2,-1).
6. Find the constants g, f and c of the circles $x^2+y^2-2x+3y-7=0$ and $x^2+y^2+4x-6y+2=0$
7. Show that the circles $x^2+y^2-8x+6y-23=0$ and $x^2+y^2-2x-5y+16=0$ are orthogonal
8. Evaluate the following:

   (i) $\underset{x\to 0}{\text{Lt}}\ \frac{x^2+3x}{x^2+2x}$

   (ii) $\underset{x\to 1}{\text{Lt}}\ \frac{x^{7/3}-1}{x-1}$

   (iii) $\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin 3\theta}{\theta}$

   (iv) $\underset{x\to 0}{\text{Lt}}\ \frac{x^4-2^4}{x-2}$

9. Differentiate the following with respect to x:

   (i) $y = x^3+\frac{2}{x^2}-\frac{1}{x}+\frac{3}{2}$

   (ii) $y=(x+1)(x-2)$

(iii) $y = \sin x\, e^{x}$

(iv) $y = (x+3)\tan x$

(v) $y = \sqrt{x}\log x$

(vi) $y = e^{x} \big/ \sin x$

(vii) $y = \sin x \quad \cos x$

(viii) $y = \dfrac{x-7}{x+3}$

PART - B

1. Find the equation of the concentric circle with the circle $x^2 + y^2 + 3x - 7y + 1 = 0$ and having radius 5 Units.

2. Show that the following circles touch each other.

   (i) $x^2 + y^2 - 2x + 6y + 6 = 0$ and $x^2 + y^2 - 5x + 6y + 15 = 0$

   (ii) $x^2 + y^2 + 2x - 4y - 3 = 0$ and $x^2 + y^2 - 8x + 6y + 7 = 0$

   (iii) $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$

3. Prove that the two circles $x^2 + y^2 - 25 = 0$ and $x^2 + y^2 - 18x + 24y + 125 = 0$ touch each other.

4. Find the equation of the circle which passes through the points (4,-1) and (6,5) and orthogonal with the circle $x^2 + y^2 + 8x - 4y - 23 = 0$

5. Find the equation of the circle through the point (1,1) and cuts orthogonally each of the circles $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$

6. **Evaluate the following:**

   (i) $\underset{x \to 3}{\text{Lt}} \dfrac{x^5 - 3^5}{x^2 - 3^2}$ (ii) $\underset{\theta \to 0}{\text{Lt}} \dfrac{5\sin 6\theta}{3\text{Sin}2\theta}$

   (iii) $\underset{x \to 1}{\text{Lt}} \dfrac{x^2 - 3x + 2}{x^2 + 5x - 6}$ (iv) $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \text{Cos}x}{x\text{Sin}x}$

7. Differentiate the following:

(i) $y=(3x^2+2x+1)e^x\tan x$

(ii) $y=(2x+1)(3x-7)(4x+5)$

(iii) $y=x^3\cot x\log x$

(iv) $y=e^x\sqrt{x}\,\operatorname{cosec} x$

(v) $y=(3x+1)\operatorname{cosec} x\ \sec x$

(vi) $y=\dfrac{1-\cos x}{1+\sin x}$

(vii) $y=\dfrac{x^2\sin x}{2x+1}$

(viii) $y=\dfrac{2e^x+\tan x}{e^x+1}$

(ix) $y=\dfrac{(x+2)\,(x+1)}{(x-2)\,(x-1)}$

(x) $y=\dfrac{1}{\sec x\tan x}$

ANSWER
PART - A

5.) K=-4

$g_1=-1 \quad g_2=2$

6.) $f_1=\dfrac{3}{2} \qquad f_2=-3$

$c_1=-7 \quad c_2=2$

8.) (i) $\dfrac{3}{2}$ (ii) $\dfrac{7}{3}$ (iii) 3 (iv) 32

9.) (i) $3x^2-\dfrac{4}{x^3}+\dfrac{1}{x^2}$

(ii) 2x-1

(iii) $\sin x\,e^x+\cos x\,e^x$

(iv) $(x+3)\sec^2 x + \tan x$

(v) $\sqrt{x}/x + \dfrac{\log x}{2\sqrt{x}}$

(vi) $\dfrac{\sin x e^x - e^x \cos x}{\sin^2 x}$

(vii) $\cos^2 x - \sin^2 x$

(viii) $\dfrac{10}{(x+3)^2}$

PART - B

1. $4x^2 + 4y^2 + 12x - 28y - \dfrac{21}{2} = 0$
2. (i) Internally (ii) Externally (iii) Externally
4. $x^2 + y^2 - 6x - 8y + 15 = 0$
5. $3x^2 + 3y^2 - 14x + 23y - 15 = 0$
6. (i) $\dfrac{405}{6}$ (ii) 5 (iii) $-\dfrac{1}{7}$ (iv) $\dfrac{1}{2}$
7. (i) $(3x^2+2x+1)e^x \sec^2 x + (3x^2+2x+1)e^x \tan x + (6x+2)e^x \tan x$

   (ii) $(2x+1)(3x-7)4 + (2x+1)(4x+5)3 + (3x-7)(4x+5)2$

   (iii) $x^3 \mathrm{Cot}x(\frac{1}{x}) + x^3 \log x(-\mathrm{coec}^2 x) + \mathrm{Cot}x \log x(3x^2)$

   (iv) $e^x\sqrt{x}(-\operatorname{cosec} x \ \cot x) + e^x \operatorname{cosec} x\left(\dfrac{1}{2\sqrt{x}}\right)$

   (v) $(3x+1)\cos x(\sec x \ \tan x) + (3x+1)\sec x(-\sin x) + 3\cos x \ \sec x$

   (vi) $\dfrac{(1+\sin x)\sin x - (1+\sin x)\mathrm{Cos}x}{(1+\sin x)^2}$

   (vii) $\dfrac{(2x+1)(x^2\cos x + 2x\sin x) - 2x^2 \sin x}{(2x+1)^2}$

(viii) $\dfrac{(e^x+1)\left[2e^x+\sec^2 x\right]-\left(2e^x+\tan x\right)(e^x)}{(e^x+1)^2}$

(ix) $\dfrac{\left(x^2-3x+2\right)\left[2x+3\right]-\left(x^2+3x+2\right)\left(2x-3\right)}{(x^2-3x+2)^2}$

(x) $-\cos x \operatorname{cosec}^2 x-\cot x \sin x$

UNIT- III

DIFFERENTIATION METHODS

3.1 Differentiation of function of functions and Implicit functions. Simple Problems.

3.2 Differentiation of inverse trigonometric functions and parametric functions Simple problems.

3.3 Successive differentiation up to second order (parametric form not included) Definition of differential equation, formation of differential equation. Simple Problems.

DIFFERENTIATION METHODS

3.1 DIFFERENTIATION OF FUNCTION OF FUNCTIONS

Function of Functions Rule:

If 'y' is a function of 'u' and 'u' is a function 'x' then

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ It is called Function of function Rule. This rule can be extended.

Chain Rule:

If 'y' is a function of 'u' and 'u' is a function of 'v' and 'v' is a function of 'x' then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

3.1 WORKED EXAMPLES

PART - A

Find $\frac{dy}{dx}$ if

1) $y = \text{Sin}(3x+4)$

2) $y = \text{Cos}(2x+3)$

3) $y = \tan(4x+3)$

4) $y = \log(\sec x + \tan x)$

5) $y = \tan(x^2+7x+3)^{11}$

Solution:

1.) $y = \mathrm{Sin}(3x+4)$

Put u= 3x+4

$$\begin{array}{l|l} \therefore y = \sin u & u = 3x+4 \\ \dfrac{dy}{du} = \mathrm{Cos}\ u & \dfrac{du}{dx} = 3 \end{array}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \mathrm{Cos} u(3) = 3\mathrm{Cos}(3x+4)$$

2.) $y = \mathrm{Cos}(2x+3)$

Put $u = 2x+3$

$$\begin{array}{l|l} \therefore y = \mathrm{Cos}\, u & u = 2x+3 \\ \dfrac{dy}{du} = -\mathrm{Sin}\ u & \dfrac{du}{dx} = 2 \end{array}$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = (-\sin u)2$$

$$= -2\mathrm{Sin}(2x+3)$$

3.) $y = \tan(4x+3)$

Put $u = 4x+3$

$$\begin{array}{l|l} \therefore y = \tan u & u = 4x+3 \\ \dfrac{dy}{du} = \mathrm{Sec}^2 u & \dfrac{du}{dx} = 4 \end{array}$$

$$\therefore \frac{dy}{dx} = 4\sec^2 u = 4\sec^2(4x+3)$$

4.) $y = \log(\sec x + \tan x)$

Put $u = \sec x + \tan x$

$$\begin{array}{l|l} \therefore y = \log u & u = \sec x + \tan x \\ \dfrac{dy}{du} = \dfrac{1}{u} & \dfrac{du}{dx} = \sec x \tan x + \sec^2 x \\ & = \sec x(\tan x + \sec x) \end{array}$$

$$\frac{dy}{dx} = \frac{1}{u}\left[\sec x(\sec x + \tan x)\right] = \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

5.) $y = \left(x^2 + 7x + 3\right)^{11}$

Put $u = x^2 + 7x + 3$

$\therefore y = u^{11}$ | $u = x^2 + 7x + 3$

$\frac{dy}{du} = 11\, u^{10}$ | $\frac{du}{dx} = 2x + 7$

$$\frac{dy}{dx} = 11\, u^{10}\left(2x + 7\right) = 11\, \left(x^2 + 7x + 3\right)^{10}\left(2x + 7\right)$$

PART - B

Differentiate the following w.r.t.to 'x'

1.) $\text{Sin}\left(x^2 + 1\right)$

2.) e^{Sinx}

3.) $\text{Sin}^3 x$

4.) logsinx

5.) $e^{\text{Sin}^2 x}$

6.) log(sin5x)

7.) $e^{\tan^{-1} 2x}$

8.) $\sin^4 3x$

9.) $\log\left(\sec^2 x\right)$

10.) $\cos\left(e^{5x}\right)$

Solution:

1.) Let $y = \sin\left(x^2 + 1\right)$

Put $u = x^2 + 1$

$\therefore y = \sin u$ | $u = x^2 + 1$

$\frac{dy}{du} = \cos u$ | $\frac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \cos u(2x)$$

$$= 2x\text{Cos}(x^2 + 1)$$

2.) $y = e^{\text{Sinx}}$

Put $u = \sin x$

$\therefore y = e^u$ | $u = \sin x$

$\frac{dy}{du} = e^u$ | $\frac{du}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u.\cos x = e^{\sin x}.\cos x$$

3.) $y = \sin^3 x$

Put $u = \text{Sin}x$

$$\therefore y = u^3 \quad\Big|\quad u = \sin x$$

$$\frac{dy}{du} = 3u^2 \quad\Big|\quad \frac{du}{dx} = \cos x$$

$$\frac{dy}{dx} = 3u^2.\text{Cos}x = 3\,\text{Sin}^2 x.\text{Cos}x$$

4.) $y = \log(\sin x)$

Put $u = \text{Sin}x$

$$\therefore y = \log u \quad\Big|\quad u = \sin x$$

$$\frac{dy}{du} = \frac{1}{u} \quad\Big|\quad \frac{du}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{1}{u}\text{Cos}x = \frac{\text{Cos}x}{\sin x} = \text{Cot}x$$

5.) $y = e^{\sin^2 x}$

Put $u = \text{Sin}^2 x$

$y = e^u$

$$\frac{dy}{dx} = e^u.\frac{du}{dx} = e^u.\frac{d}{dx}\left(\sin^2 x\right)$$

$$= e^{\sin^2 x}\left(2\sin x \quad \text{Cos}x\right)$$

6.) $y = \log(\sin 5x)$

$$\frac{dy}{dx} = \frac{1}{\sin 5x}\frac{d}{dx}\left(\sin 5x\right)$$

$$= \frac{1}{\sin 5x}\left(5\cos 5x\right) = 5\cot 5x$$

7.) $y = e^{\tan^{-1}(2x)}$

$$\frac{dy}{dx} = e^{\tan^{-1} 2x}\frac{d}{dx}\left(\tan^{-1} 2x\right)$$

$$= e^{\tan^{-1} 2x}.\frac{1}{1+(2x)^2}(2)$$

$$= \frac{2e^{\tan^{-1} 2x}}{1+4x^2}$$

8.) $y = \text{Sin}^4 3x = (\sin 3x)^4$

$$\frac{dy}{dx} = 4(\sin 3x)^3 \frac{d}{dx}(\sin 3x)$$

$$= 4\sin^3 3x(3 \quad \text{Cos}3x)$$

$$= 12\sin^3 3x \ \text{Cos}3x$$

9.) $Y = \log\left(\sec^2 x\right)$

$$\frac{dy}{dx} = \frac{1}{\sec^2 x} \ \frac{d}{dx}(\sec^2 x)$$

$$= \frac{1}{\sec^2 x}(2\sec x)(\sec x \ \tan x) = 2\tan x$$

10.) $y = \text{Cos}\left(e^{5x}\right)$

$$\frac{dy}{dx} = -\sin\left(e^{5x}\right)\frac{d}{dx}\left(e^{5x}\right)$$

$$= -\sin\left(e^{5x}\right).5e^{5x}$$

$$= -5e^{5x}\sin\left(e^{5x}\right)$$

3.1.2. Differentiation of implicit functions

If the variables x and y are connected by the relation of the form f(x,y)= 0 and it is not possible to express y as a function of x in the form y=f(x) then y is said to be an implicit function of x. To find $\frac{dy}{dx}$ in this case, we differentiate both sides of the given relation with respect to x. keeping in mind that derivative of $\phi(y)$

w.r.t 'x' as $\frac{d\phi}{dy}.\frac{dy}{dx}$ i.e $\frac{d}{dx}(\phi(y)) = \frac{d\phi}{dy}.\frac{dy}{dx}$

For Example $\frac{d}{dx}(\sin y) = \text{Cosy}\frac{dy}{dx}$ and $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$

WORKED EXAMPLES
PART – A

Find $\frac{dy}{dx}$ for the following functions

1) $xy = c^2$
2) $y = \cos(x + y)$
3) $y^2 = 4ax$
4) $x^2 + y^2 = a^2$
5) $xy^2 = k$

Solution:

1) $xy = c^2$

Differentiate both sides w.r.t 'x'

$$x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

2) $y = \cos(x + y)$

Differentiate both sides w.r.t 'x'

$$\frac{dy}{dx} = -\sin(x+y)\left(1+\frac{dy}{dx}\right)$$

$$= -\sin(x+y) - \sin(x+y)\frac{dy}{dx}$$

$$\frac{dy}{dx} + \sin(x+y)\frac{dy}{dx} = -\sin(x+y)$$

$$[1+\sin(x+y)]\frac{dy}{dx} = -\sin(x+y)$$

$$\frac{dy}{dx} = -\frac{\sin(x+y)}{1+\sin(x+y)}$$

3) $y^2 = 4ax$

Differentiate both sides w.r.t 'x'

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

4) $x^2 + y^2 = a^2$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

5) $xy^2 = k$

$$x\ \ 2y\frac{dy}{dx} + y^2 = 0$$

$$2xy\frac{dy}{dx} = -y^2$$

$$\frac{dy}{dx} = \frac{-y^2}{2xy}$$

$$= \frac{-y}{2x}$$

PART - B

Find $\frac{dy}{dx}$ of the following

1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

2) $x^2 + y^2 + 2gx + 2fy + c = 0$

3) $x^3 + y^3 = 3axy$

4) $ax^2 + 2hxy + by^2 = 0$

5) $y = x \quad \sin(a + y)$

Solution:

1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Differentiate both sides w.r.t 'x'

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{2x}{a^2}.\frac{b^2}{2y}$$

$$= -\frac{b^2x}{a^2y}$$

2) $x^2 + y^2 + 2gx + 2fy + c = 0$

Differentiate both sides w.r.t 'x'

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$(2y + 2f)\frac{dy}{dx} = -2x - 2g$$

$$\frac{dy}{dx} = -\frac{2(x+g)}{2(y+f)}$$

$$= -\left(\frac{x+g}{y+f}\right)$$

3) $x^3 + y^3 = 3axy$

Differentiate both sides w.r.t 'x'

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

$$= 3ax\frac{dy}{dx} + 3ay$$

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$$

$$(3y^2 - 3ax)\frac{dy}{dx} = 3ay - 3x^2$$

$$\frac{dy}{dx} = \frac{3(ay - x^2)}{3(y^2 - ax)}$$

$$= \frac{ay - x^2}{y^2 - ax}$$

4) $ax^2 + 2hxy + by^2 = 0$

Differentiate both sides w.r.t 'x'

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0$$

$$2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$$

$$(2hx + 2by)\frac{dy}{dx} = -2ax - 2hy$$

$$\frac{dy}{dx} = \frac{-2ax - 2hy}{2hx + 2by}$$

$$= -\frac{(ax + hy)}{hx + by}$$

5) $y = x\sin(a + y)$

Differentiate both sides w.r.t 'x'

$$\frac{dy}{dx} = x\cos(a + y)\frac{dy}{dx} + \sin(a + y)$$

$$\frac{dy}{dx} - x\cos(a + y)\frac{dy}{dx} = \sin(a + y)$$

$$[1 - x\cos(a + y)]\frac{dy}{dx} = \sin(a + y)$$

$$\frac{dy}{dx} = \frac{\sin(a + y)}{1 - x\cos(a + y)}$$

3.2.1. Differentiation of Inverse Trigonometric Functions

If x = siny then

$y = \sin^{-1}x$

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$ are inverse trigonometric functions.

Example: (1)

If $y = \sin^{-1}x$, find $\frac{dy}{dx}$

$$y = \sin^{-1}x$$

$$\therefore \sin y = x$$

Differentiate both sides w.r.t 'x'

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{\cos^2 y}} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Example: (2)

Find the differentiation of $\cos^{-1}x$

Let $y = \cos^{-1}x$

$\therefore \cos y = x$

Differentiate both sides w.r.t 'x'

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{\sin^2 y}} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$$

Example: (3)

Find the differentiation of $\tan^{-1}x$.

Let $y = \tan^{-1}x$

$\therefore \tan y = x$

Differentiate both sides w.r.t 'x'

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Example: (4)

Find the differentiation of $\cot^{-1}x$.

Let $y = \cot^{-1}x$

$\therefore \cot y = x$

Differentiate both sides w.r.t 'x'

$$\therefore -\operatorname{cosec}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}$$

Example: (5)

(Differentiation of $\sec^{-1}x$)

Let $y = \sec^{-1}x$

$\therefore \sec y = x$

Differentiate both sides w.r.t 'x'

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2 - 1}}$$

Example: (6)

Differentiation of $\mathrm{cosec}^{-1}x$

Let $y = \mathrm{cosec}^{-1}x$

$\therefore \mathrm{cosec}\, y = x$

Differentiate both sides w.r.t 'x'

$$-\mathrm{cosec}\, y \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\mathrm{cosec}\, y \cot y} = -\frac{1}{\mathrm{cosec}\, y\sqrt{\mathrm{cosec}^2 y - 1}} = -\frac{1}{x\sqrt{x^2 - 1}}$$

FORMULA

1) $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1 - x^2}}$

2) $\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1 - x^2}}$

3) $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1 + x^2}$

4) $\frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1 + x^2}$

5) $\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{x^2 - 1}}$

6) $\frac{d}{dx}\left(\mathrm{cosec}^{-1}x\right) = -\frac{1}{x\sqrt{x^2 - 1}}$

WORKED EXAMPLES

PART - A

Differentiate the following w.r.t 'x'

(1) $\sin^{-1}\left(\sqrt{x}\right)$ (2) $\cos^{-1}\left(\sqrt{x}\right)$ (3) $\cot^{-1}\left(\frac{1}{x}\right)$

(4) $\left(\sin^{-1}x\right)^2$ (5) $x^2\sin^{-1}x$

(6) $\sqrt{1-x^2}\sin^{-1}x$ (7) $e^x\tan^{-1}x$ (8) $\tan^{-1}\left(\sqrt{x}\right)$

Solution:

1) $y=\sin^{-1}\left(\sqrt{x}\right)$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}\sqrt{1-x}}=\frac{1}{2\sqrt{x(1-x)}}$$

2) $y=\cos^{-1}\left(\sqrt{x}\right)$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\left(\sqrt{x}\right)=-\frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}=-\frac{1}{2\sqrt{x(1-x)}}$$

3) $y=\cot^{-1}\left(\frac{1}{x}\right)$

$$\frac{dy}{dx}=-\frac{1}{1+\left(\frac{1}{x}\right)^2}\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)$$

$$\left(-\frac{x^2}{x^2+1}\right)\left(-\frac{1}{x^2}\right)=\frac{1}{x^2+1}$$

4) $y=\left(\sin^{-1}x\right)^2$

$$\frac{dy}{dx}=2\sin^{-1}x\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$=2\sin^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right)=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

5) $y = x^2 \sin^{-1} x$

$$\frac{dy}{dx} = x^2\left(\frac{1}{\sqrt{1-x^2}}\right) + 2x\sin^{-1} x$$

$$= \frac{x^2}{\sqrt{1-x^2}} + 2x\sin^{-1} x$$

6) $y = \sqrt{1-x^2}\,\sin^{-1} x$

$$\frac{dy}{dx} = \sqrt{1-x^2}\left(\frac{1}{\sqrt{1-x^2}}\right) + \sin^{-1} x\,\frac{1}{2\sqrt{1-x^2}}(-2x) = 1 - \frac{x\sin^{-1} x}{\sqrt{1-x^2}}$$

7) $y = e^x \tan^{-1} x$

$$\frac{dy}{dx} = e^x\left(\frac{1}{1+x^2}\right) + \tan^{-1} x\left(e^x\right)$$

$$= e^x\left[\frac{1}{1+x^2} + \tan^{-1} x\right]$$

8) $y = \tan^{-1}\left(\sqrt{x}\right)$

$$\frac{dy}{dx} = \frac{1}{1+\left(\sqrt{x}\right)^2}\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{1+x}\,\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

PART - B

Differentiate the following w.r.t 'x'

1) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ 2) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ 3) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

1.) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$ $\therefore \theta = \tan^{-1} x$

$$\therefore y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \quad \left(\therefore \sin 2A = \frac{2\sin A}{1+\sin^2 A}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$
$$= 2\theta$$
$$y = 2\tan^{-1} x$$
$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

2.) Let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Put $x = \tan\theta \qquad \therefore \theta = \tan^{-1}x$

$$\therefore y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$
$$= \tan^{-1}(\tan 2\theta) \qquad \left[\because \tan 2A = \frac{2\tan A}{1-\tan^2 A}\right]$$
$$= 2\theta$$
$$y = 2\tan^{-1} x$$
$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

3.) Let $y = \text{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta \qquad \therefore \theta = \tan^{-1}x$

$$\therefore y = \text{Cos}^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$
$$= \text{Cos}^{-1}(\text{Cos}2\theta) \qquad \left[\text{Cos}2A = \frac{1-\tan^2 A}{1+\tan^2 A}\right]$$
$$= 2\theta$$
$$\therefore y = 2\tan^{-1} x$$
$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

3.2.2. Differentiation of parametric function:

Sometimes 'x' and 'y' are functions of a third variable called a parameter without eliminating the parameter, we can find $\frac{dy}{dx}$ as follows.

Let $x = f(\theta)$ and $y = F(\theta)$ so that 'θ' is a parameter.

$$\frac{dy}{dx} = \frac{dy}{d\theta} . \frac{d\theta}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

WORKED EXAMPLES

PART-A

1.) If $x = a \ \text{Cos}\ \theta$ and $y = b\ \text{Sin}\ \theta$, find $\frac{dy}{dx}$

Solution:

$$\frac{dx}{d\theta} = -a\sin\theta$$

$$\frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\ \theta$$

2.) Find $\frac{dy}{dx}$ if $x = at^2$ and $y = 2at$

Solution:

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

3.) Find $\frac{dy}{dx}$ if $x = a\sec\theta$ and $y = b\tan\theta$

Solution:

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

$$= \frac{b\left(\frac{1}{Cos\theta}\right)}{a\left(\frac{\sin\theta}{Cos\theta}\right)} = \frac{b}{a\sin\theta} = \frac{b}{a}Cosec\,\theta$$

4.) Find $\frac{dy}{dx}$ if x= ct and $y = \frac{c}{t}$

Solution:

$$\frac{dx}{dt} = c$$

$$\frac{dy}{dt} = -\frac{c}{t^2}$$

$$\frac{dy}{dx} = -\frac{-c/t^2}{c} = -\frac{1}{t^2}$$

PART - B

1.) Find $\frac{dy}{dx}$ if x= cost + tsint and y=sint- tcost

Solution:

$$\frac{dx}{dt} = -\sin t + t\cos t + \sin t = t\cos t$$

$$\frac{dy}{dt} = \cos t + t\sin t - \cos t = t\sin t$$

$$\frac{dy}{dx} = \frac{t\sin t}{t\cos t} = \tan t$$

2.) Find $\frac{dy}{dx}$ if $x = \sqrt{\sin 2t}$ and $y = \sqrt{\cos 2t}$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{\sin 2t}} \cos 2t(2) = \frac{\cos 2t}{\sqrt{\sin 2t}}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{\cos 2t}} (-\sin 2t)2 = \frac{-\sin 2t}{\sqrt{\cos 2t}}$$

$$\frac{dy}{dx} = \frac{\left(\frac{-\sin 2t}{\sqrt{\cos 2t}}\right)}{\left(\frac{\cos 2t}{\sqrt{\sin 2t}}\right)} = -\frac{\sin 2t\sqrt{\sin 2t}}{\cos 2t\sqrt{\cos 2t}}$$

$$= -\frac{(\sin 2t)^{3/2}}{(\cos 2t)^{3/2}} = -(\tan 2t)^{3/2}$$

3.) Find $\frac{dy}{dx}$ if $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

Solution:

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = a(0 + \sin\theta) = a(\sin\theta)$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} = \tan\frac{\theta}{2}$$

4.) Find $\frac{dy}{dx}$ if $y = \log(\sec\theta + \tan\theta)$ and $x = \sec\theta$

Solution:

$$\frac{dy}{d\theta} = \frac{1}{\sec\theta + \tan\theta}\left(\sec\theta\tan\theta + \sec^2\theta\right)$$

$$= \frac{\sec\theta(\tan\theta + \sec\theta)}{\sec\theta + \tan\theta} = \sec\theta$$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta$$

$$\frac{dy}{dx} = \frac{\sec\theta}{\sec\theta\tan\theta} = \frac{1}{\tan\theta} = \cot\theta.$$

3.3.1 Successive differentiation:

If $y = f(x)$, then $\frac{dy}{dx}$, the derivative of y w.r. to x is a function of x and can be differentiated once again. To fix the idea, we shall call $\frac{dy}{dx}$ as the first order derivative of y with respect to x and the derivative of $\frac{dy}{dx}$ w.r. to 'x' as a second order derivative of y w.r. to x and will be denoted by $\frac{d^2y}{dx^2}$. Similarly the derivative of $\frac{d^2y}{dx^2}$ w.r.t 'x' will be called third order derivative and will be denoted by $\frac{d^3y}{dx^3}$ and so on.

Note: Alternative notations for

$$\frac{dy}{dx} = y_1 = y' = f'(x) = D(y)$$

$$\frac{d^2y}{dx^2} = y_2 = y'' = f''(x) = D^2(y)$$

$$\frac{d^ny}{dx^n} = y_n = y^{(n)} = f^{(n)}(x) = D^n(y)$$

WORKED EXAMPLES

PART - A

1.) If $y = \tan x$, find $\frac{d^2y}{dx^2}$

$$y = \tan x,$$

$$\frac{dy}{dx} = \sec^2 x$$

$$\frac{d^2y}{dx^2} = 2\sec x(\sec x \tan x)$$

$$= 2 \sec^2 x \tan x$$

2.) If $y = \log(\sin x)$, find y_2

$$y_1 = \frac{1}{\sin x}\cos x = \cot x$$

$$y_2 = -\operatorname{cosec}^2 x.$$

3.) If $y = a\cos x + b\sin x$, find y_2

$$y_1 = -a\sin x + b\cos x$$

$$y_2 = -a\cos x - b\sin x.$$

4.) If $y = Ae^{3x} + Be^{-5x}$, find $D^2(y)$

$$D(y) = 3 A e^{3x} - 5 B e^{-5x}$$

$$D^2(y) = 9 A e^{3x} + 25 B e^{-5x}$$

5.) If $y = \frac{1}{x}$, find y_2

$$y_1 = -\frac{1}{x^2}$$

$$y_2 = \frac{2}{x^3}$$

PART - B

1.) If $y = x^2 \sin x$, prove that $x^2 y_2 - 4xy_1 + (x^2+6)y = 0$

Solution:

$$y = x^2 \sin x$$

$$y_1 = x^2(\cos x) + 2x\sin x = x^2\cos x + 2x\sin x$$

$$y_2 = x^2(-\sin x) + 2x\cos x + 2x\cos x + 2\sin x$$

$$= -x^2\sin x + 4x\cos x + 2\sin x$$

$$\therefore x^2y_2 - 4xy_1 + (x^2+6)y = x^2\left[-x^2\sin x + 4x\cos x + 2\sin x\right]$$

$$- 4x(x^2\cos x + 2x\sin x) + (x^2+6)x^2\sin x$$

$$= -x^4\sin x + 4x^3\cos x + 2x^2\sin x - 4x^3\cos x$$

$$-8x^2\sin x + x^4\sin x + 6x^2\sin x$$

$$= 0$$

2.) If $y = a\cos\log x + b\sin\log x$, show that $x^2y_2 + xy_1 + y = 0$,

Solution:

$$y = a\cos\log x + b\sin\log x$$

$$y_1 = -a\sin\log x\left(\frac{1}{x}\right) + b\cos\log x\left(\frac{1}{x}\right)$$

Multiply by x on both sides

$$\therefore xy_1 = -a\sin\log x + b\cos\log x$$

Differentiate both sides w.r.to x

$$xy_2 + y_1 = -a\cos\log x\left(\frac{1}{x}\right) - b\sin\log x\left(\frac{1}{x}\right)$$

Again multiply by x on both sides

$$\therefore x^2y_2 + xy_1 = -a\cos\log x - b\sin\log x$$

$$= -(a\cos\log x + b\sin\log x)$$

$$= -y$$

$$\therefore x^2y_2 + xy_1 + y = 0$$

3.) If $y = e^{\sin^{-1}x}$, prove that $(1-x^2)y_2 - xy_1 - y = 0$

$y = e^{\sin^{-1}x}$

$y_1 = e^{\sin^{-1}x} . \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\ y_1 = e^{\sin^{-1}x}$

Differentiate both sides w.r.t x

$\sqrt{1-x^2}\ y_2 + y_1 \frac{1}{2\sqrt{1-x^2}}(-2x) = e^{\sin^{-1}x} \frac{1}{\sqrt{1-x^2}}$

Multiply by $\sqrt{1-x^2}$ on both sides

$(1-x^2)y_2 - xy_1 = y$

4.) If $y = (\tan^{-1}x)^2$, prove that $(1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$

Solution:

$y = (\tan^{-1}x)^2$

$y_1 = 2\tan^{-1}x\left(\frac{1}{1+x^2}\right)$

Multiply by $(1+x^2)$ on both sides

$(1+x^2)y_1 = 2\tan^{-1}x$

Differentiate both sides w.r.t.x

$(1+x^2)y_2 + y_1(2x) = 2\frac{1}{1+x^2}$

Multiply by $(1+x^2)$ on both sides

$(1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$

3.3.2 Formation of differential equation:

Definition: An equation involving differential coefficients is called differential equation.

Examples:

1.) $\frac{dy}{dx} = 2xy$

2.) $\frac{dy}{dx} + 2xy = x^2$

3.) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

4.) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(1 + \frac{dy}{dx}\right)^3 = 0$

Order of the differential equation:

The order of the differential equation is the order of the highest derivative appearing in the equation, after removing the radical sign and fraction involved in the equation.

e.g: 1.) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$ (order is 2)

2.) $\frac{d^3y}{dx^3} + 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$ (order is 3)

Degree of the differential equation:

The degree of the differential equation is the power (or) degree of the highest derivative occuring in the differential equation.

Example:

1.) $5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 5y = e^x$ (degree 1)

2.) $7\left(\frac{d^2y}{dx^2}\right)^2 + 5\left(\frac{dy}{dx}\right)^5 + 7y = \sin x$ (degree 2)

WORKED EXAMPLES
PART - A

1.) If $xy = c^2$, form the differential equation by eliminating the Constant 'c'

$$xy = c^2$$

$$x\frac{dy}{dx} + y = 0$$

PART - B

1.) If y= A Cos 2x+ Bsin2x, form the differential equation by eliminating the constants 'A' and 'B'.

y=A Cos 2x + B sin 2x

$$\frac{dy}{dx} = -2A\sin 2x + 2B\text{Cos}2x$$

$$\frac{d^2y}{dx^2} = -4A\text{Cos}2x - 4B\sin 2x$$

$$= -4\left[A\text{Cos}2x + B\sin 2x\right]$$

$$= -4y$$

$\frac{d^2y}{dx^2} + 4y = 0$ is the required differential equation

2) Form the differential equation by eliminating the constant 'a' and 'b' from the equation.

$$xy = ae^x + be^{-x}$$

Solution:

Given $xy = ae^x + be^{-x}$

Differentiate both sides w.r.to x

$$xy_1 + y = ae^x - be^{-x}$$

Differentiate again w.r. to x,

$$x\,y_2 + y_1 + y_1 = ae^x + be^{-x}$$

$$x\,y_2 + 2y_1 = xy$$

$$x\,y_2 + 2y_1 - xy = 0$$

is the required differential equation.

EXERCISE
PART - A

I. Differentiate the following w.r. to ‘x’

1.) $(3x+5)^{10}$ (2.) $\sin^2(5x+3)$ (3.) $e^{\cos^3 x}$

4.) $(\tan^{-1}3x)^2$ (5.) $e^{\sin^{-1}2x}$ (6.) $\log(\tan^2 x)$

II. Find $\frac{dy}{dx}$ if

1.) $x^2+y^2=0$ (2.) $x \sin y=0$

3.) $xy=k$ (4.) $xy^2=x+y$

III. 1.) $\sin^{-1}5x$ (2.) $\sin^{-1}(\cos x)$

3.) $\sin^{-1}3x$ (4.) $(\tan^{-1}4x)^3$

PART - B

I) Differentiate the following w.r.t ‘x’

1.) $e^{5x}\cos 2x\ \sin 3x$ 2.) $(x^2+1)\text{Cos}5x$

3.) $e^{\sin^{-1}x}\cos 6x$ 4.) $e^{\tan^{-1}2x}\tan 5x$

5.) $x^2\tan x$ 6.) $x\text{Cos}^{-1}(x)$

7.) $(1+x^2)\tan^{-1}x$ 8.) $\sqrt{x}\sin^{-1}x$

II) Find $\frac{dy}{dx}$ of the following

1.) $2x^2+6xy+y^2=1$

2.) $x^2\sin y=C$

3.) $y=a+xe^y$

4.) $x^2+3xy+2y^2=4$

5.) $x^m y^n=a^{m+n}$

III) Differentiate the following w.r.t 'x'

1.) $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

2.) $y = \sin^{-1}\left(\dfrac{x}{\sqrt{x^2 + a^2}}\right)$

3.) $y = \sin^{-1}(3x - 4x^3)$

4.) $y = \cos^{-1}(4x^3 - 3x)$

5.) $y = \dfrac{\sqrt{1 - x^2} - 4}{x}$

6.) $y = \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$

7.) $y = \sec^{-1}\left(\dfrac{1}{2x^2 - 1}\right)$

8.) $y = \sin^{-1}\left(\dfrac{1}{1 - 2x^2}\right)$

IV) Find $\dfrac{dy}{dx}$ if

1.) $x = 3\sin t - \sin^3 t$ and $y = \cos t - \cos^3 t$

2.) $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$

3.) $x = \cos^3\theta$ and $y = \sin^3\theta$

V) 1.) If $y = x^2 \cos x$, Prove that $(1 - x^2)y_2 - 4x\, y_1 + (x^2 + 6)y = 0$

2.) If $y = (\sin^{-1} x)^2$, Show that $(1 - x^2)y_2 - x\, y_1 - 2 = 0$

3.) If $y = e^{\tan^{-1} x}$, Prove that $(1 - x^2)y_2 + (2x - 1)y_1 = 0$

4.) If $y = \log\left[x + \sqrt{x^2 + 1}\right]^2$, Show that $(1 + x^2)y_2 + xy_1 = 2$

5.) If $y = e^{a\cos^{-1} x}$, Prove that $(1 - x^2)y_2 - x\, y_1 - a^2 y = 0$

ANSWER
PART - A

I 1.) $30(3X + 5)^9$

2.) $10\sin(5x + 3)\cos(5x + 3)$

3.) $-3e^{\cos 3x}\mathrm{Cos}^2 x \sin x$

4.) $\dfrac{6\tan^{-1} 3x}{1 + 9x^2}$

5.) $\dfrac{2e^{\sin^{-1}(2x)}}{\sqrt{1 - 4x^2}}$

6.) Cosec x secx

II. 1.) $\frac{-x}{y}$ 2.) $\frac{-\tan y}{x}$ 3.) $\frac{-y}{x}$ 4.) $\frac{1-y^2}{2xy-1}$

III. 1.) $\frac{-5}{\sqrt{1-25x^2}}$ 2.) -1 3.) $\frac{1}{x\sqrt{9x^2-1}}$

4.) $\frac{12\left(\tan^{-1}4x\right)^2}{1+16x^2}$

PART - B

I 1.) $e^{5x}[3\cos 2x\cos 3x]-2\sin 2x\sin 3x+5\cos 2x\sin 3x$

2.) $2x\cos 5x-5\left(x^2+1\right)\sin 5x$

3.) $e^{\sin-1x}\left[-6\sin 6x+\frac{\cos 6x}{\sqrt{1-x^2}}\right]$

4.) $e^{\tan^{-1}2x}\left[5\sec^2 5x+\frac{2\tan 5x}{1+4x^2}\right]$

5.) $\frac{x^2}{1+x^2}+2x\tan^{-1}x$ 6.) $\frac{-x}{\sqrt{1-x^2}}+\cos^{-1}x$

7.) $1+2x\tan^{-1}x$ 8.) $\sqrt{\frac{x}{1-x^2}}+\frac{\sin^{-1}\sqrt{x}}{2\sqrt{x}}$

II. 1.) $-\frac{2x+3y}{3x+y}$ 2.) $-\frac{2}{x}\tan y$ 3.) $\frac{e^y}{1-xe^y}$

4.) $-\frac{2x+3y}{3x+4y}$ 5.) $-\frac{my}{nx}$

III. 1.) $\frac{3}{1+x^2}$ 2.) $\frac{a}{a^2+x^2}$ 3.) $\frac{3}{\sqrt{1-x^2}}$ 4.) $\frac{-3}{\sqrt{1-x^2}}$

5.) $\frac{1}{2\left(1+x^2\right)}$ 6.) $\frac{2}{\sqrt{1-x^2}}$ 7.) $\frac{-2}{\sqrt{1-x^2}}$ 8.) $\frac{2}{\sqrt{1-x^2}}$

IV. 1.) $-\tan^3 t$ 2.) $\cot\frac{\theta}{2}$ 3.) $-\tan\theta$

UNIT- IV

APPLICATION OF DIFFERENTIATION–I

4.1 Derivative as a rate measure-simple Problems**.**
4.2 Velocity and Acceleration-simple Problems
4.3 Tangents and Normals-simple Problems

4.1. DERIVATIVE AS A RATE MEASURE:

Let y=f(x) is a differentiable function of x in any interval of x. Let Δx be a small increment in x and the corresponding increment in y is Δy. Then $\frac{\Delta y}{\Delta x}$ represents the average change in the interval x and x+Δx. When Δx tends to zero, the average rate of change $\frac{\Delta y}{\Delta x}$ will become nearer and nearer to the actual rate of change of y at x. Thus $\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ represents the actual rate of change of y at x.

Note: If r, v and s are radius volume and surface area, then $\frac{dr}{dt}, \frac{dv}{dt}$ and $\frac{ds}{dt}$ are their rate of change with respect to time ‘t’.

4.1 WORKED EXAMPLES

PART - B

1. The radius of a metal plate is increasing uniformly at the rate of 0.05 cm per second on heating. At what rate the area is increasing when the radius is 10cm.

Solution:

Let ‘r’ be the radius

Given : $\frac{dr}{dt} = 0.05$

To find $\frac{dA}{dt}$ when r=10.cm.

We know that area = $A = \pi r^2$

$$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Put $r = 10$ and $\frac{dr}{dt} = 0.05$.

$$\therefore \frac{dA}{dt} = 2\pi(10)(0.05) \qquad = \pi \text{ sq.cm./sec}$$

The Area is increasing at the rate of π sq.cm / sec

2. The area of a square is increasing at the rate of 4 cm² / sec. How fast its side is increasing when it is 8cm long?

Solution:

Let 'x' be the side of the square .

Given : $\frac{dA}{dt} = 4 \text{ cm}^2/\text{sec}$

To find $\frac{dx}{dt}$ when $x = 8$cm.

$$A = x^2$$

$$\therefore \frac{dA}{dt} = 2x \frac{dx}{dt}$$

$$4 = 2(8)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{4}{16} = \frac{1}{4} \text{ cm/sec.}$$

The side is increasing at the rate of $\frac{1}{4}$ cm / sec.

3. The radius of a sphere is increasing at the rate of 2cm. per second How fast its volume is increasing when the radius is 20cm?

Solution:

Let 'r' be the radius of the sphere and 'V' be the volume.

Given : $\frac{dr}{dt} = 2$ cm /sec

To find : $\dfrac{dv}{dt}$ when r = 20 cm.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt} \qquad = 4\pi(20)^2 2$$

$$= 3200\pi \text{ cm}^3/\text{sec}.$$

The volume is increasing at the rate of 3200 $\pi \text{cm}^3/\text{sec}$

4. The radius of a sphere is increasing at the rate of 3cm/sec. Find the rate at which its surface area is increasing when the radius is 10cm.

Solution:

Let ‘r ‘be the radius and ‘S’ be the surface area

Given : $\dfrac{dr}{dt} = 3$ cm /sec

To find : $\dfrac{dS}{dt}$ when $r = 10$cm

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 4\pi 2r\frac{dr}{dt} \qquad = 8\pi r\frac{dr}{dt} \qquad = 8\pi(10)3$$

$$= 240\pi \text{ cm}^2/\text{sec}$$

The surface area is increasing at the rate of $= 240\pi \text{ cm}^2/\text{sec}$

5. A balloon which remains spherical is being inflated by pumping in 10 cm^3 of gas per second Find the rate at which the radius of the balloon is increasing when the radius is 15cm.

Solution:

Let ‘r’ be the radius and ‘V’ be the volume.

Given : $\dfrac{dV}{dt} = 10 \text{ cm}^3$ /sec

To find : $\dfrac{dr}{dt}$ when $r = 15$cm

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} \qquad = 4\pi r^2 \frac{dr}{dt}$$

$$10 = 4\pi(15)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{10}{900\pi} = \frac{1}{90\pi}$$

The radius is increasing at the rate of $\frac{1}{90\pi}$ cm / sec

6. A stone thrown in to still water causes a series of concentric ripples. If the radius of the outer ripple is increasing at the rate of 2 meter / sec, how fast is the area is increasing when the outer ripple has a radius of 10 meters.

Solution:

Let ‘r ‘be the radius

Given : $\frac{dr}{dt} = 2$ meter /sec

To find : $\frac{dA}{dt}$ when $r = 10$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt} \qquad = \pi 2(10)2 \qquad = 40\pi m^2 / sec$$

The outer area of the ripple is increasing at the rate of $40\pi m^2 / sec$

7. The side of an equilateral triangle is increasing at the rate of 5cm/sec. Find the rate of change of its area when the side is 8cm.

Solution:

Let ‘x’ be the side of the equilateral triangle

Given : $\frac{dx}{dt} = 5$ cm /sec

To find : $\frac{dA}{dt}$ when x = 8cm

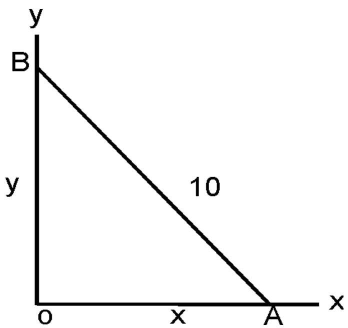
$$\text{Area} = A = \frac{\sqrt{3}}{4}x^2$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}2x\frac{dx}{dt} = \frac{\sqrt{3}}{4}2(8)5 \qquad = 20\sqrt{3}\,cm^2/sec$$

$\therefore$ The rate of change of its area is 20 $\sqrt{3}$ cm² /sec.

8. A rod 10 m long moves with its ends A and B always on the axes of x and y respectively. If A is 8 m from the origin and is moving away at the rate of 2 m per sec. Find at what rate the end B is moving.

Solution:



Let AB be the rod.

At time 't' sec the rod may be the above position.

Let OA = x and OB = y.

Given: $\frac{dx}{dt} = 2$

To find $\frac{dy}{dt}$ when x =8 m

We know that $x^2 + y^2 = 10^2$

$$\therefore y^2 = 100 - x^2$$

$$y = \sqrt{100 - x^2}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{100 - x^2}}(-2x)\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{-x}{\sqrt{100 - x^2}} \times \frac{dx}{dt}$$

$$= \frac{-8}{\sqrt{100 - 64}}(2)$$

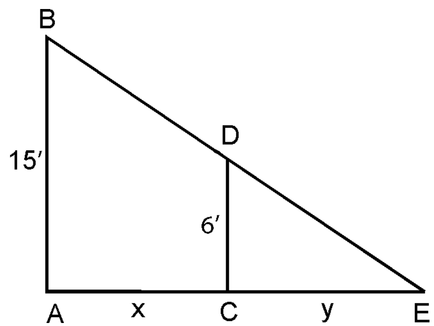
$$= \frac{-16}{6} \quad = \frac{-8}{3} \text{ m/sec}$$

The end of the rod is moving at the rate of $\frac{8}{3}$ m/sec

9.) A man of height 6 feet walks away from a lamp post of height 15 feet at a constant speed of 2 feet per second

Find the rate at which his shadow on the ground is increasing when the man is 10 feet from the lamp post.

Solution:



Let AB = Lamp post CD= man,

CE = length of the shadow

Let 'x' be the distance between the lamp post and man and 'y' be the length of the shadow.

Given: $\frac{dx}{dt} = 2$

To find: $\frac{dy}{dt}$ when x=10 feet

Here Δ ABE similar to Δ CDE

$$\therefore \frac{AE}{CE} = \frac{AB}{CD}$$

$$\frac{x+y}{y} = \frac{15}{6}$$

$6x + 6y = 15y,\quad 9y = 6x,\qquad \therefore 3y = 2x$

$$3\frac{dy}{dt} = 2\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{2}{3}\ \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{2}{3}(2) \qquad = \frac{4}{3}$$

$\therefore$ The shadow is lengthening at the rate of $\frac{4}{3}$ feet / sec

4.2 VELOCITY AND ACCELERATION

Let the distance ‘s’ be a function of time ‘t’

Velocity is the rate of change of displacement

Velocity $v = \frac{ds}{dt}$

Acceleration is the rate of change of velocity.

$\therefore$ acceleration is $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$.

Note:

1. At time t=0, we get the initial velocity.
2. When the particle comes to rest, then velocity v=0
3. When the acceleration is Zero, velocity is Uniform.

4.2 WORKED EXAMPLE

PART - A

1.) If $s = 5t^3 - 3t^2 + 10t$ find the velocity and acceleration after 't' seconds.

Solution:

$$s = 5t^3 - 3t^2 + 10t$$

$$\text{Velocity} = v = \frac{ds}{dt} = 15t^2 - 6t + 10$$

$$\text{acceleration} = a = \frac{dv}{dt} = 30t - 6$$

2.) The distance 's' travelled by a particle in 't' sec is given by $s = 4t^3 - 5t + 6$ Find the velocity at the end of 10 seconds.

Solution:

$$s = 4t^3 - 5t + 6$$

$$v = \frac{ds}{dt} = 12t^2 - 5$$

When $t = 10$, $v = 12(10)^2 - 5$

$$= 1200 - 5 = 1195\, m/sec$$

3.) The distance time formula of a moving particle is given by $s = 7t^3 - 5t^2 + 12t - 13$ Find the initial acceleration

Solution:

$$s = 7t^3 - 5t^2 + 12t - 13$$

$$v = \frac{ds}{dt} = 21t^2 - 10t + 12$$

$$\text{acceleration} = a = \frac{dv}{dt} = 42t - 10$$

Initial acceleration =42(0) -10=-10 Unit /sec^2.

4.) If the distance ‘s‘ meters travelled by a body in ‘t’ seconds is given by $s = 5t^2 - 20t + 12$. Find at what time the velocity vanishes.

Solution:

$$s = 5t^2 - 20t + 12$$

$$v = \frac{ds}{dt} = 10t - 20$$

Velocity vanishes, v=0

$\therefore\ 10t - 20 = 0,\ 10t = 20$

$\therefore\ t = 2\,secs$

5.) Find the initial velocity of the body whose distance time relation is $s = 8\cos 2t + 4\sin t$

Solution:

$$s = 8\cos 2t + 4\sin t$$

$$v = \frac{ds}{dt} = -16\sin 2t + 4\cos t$$

$$\text{Initial velocity} = -16\sin 0 + 4\cos 0$$

$$= 0 + 4 = 4$$

6.) The distance time formula of a moving particle is given by $s = ae^t + be^{-t}$. Show that the acceleration is always equal to the distance travelled.

Solution:

$$s = ae^t + be^{-t}$$

$$\text{Velocity} = v = \frac{ds}{dt} = ae^t - be^{-t}$$

$$\text{Acceleration} = a = \frac{dv}{dt} = ae^t + be^{-t}$$

$$= s.$$

$\therefore$ Acceleration = distance travelled.

PART - B

1.) The distance 's' meters at time 't' seconds travelled by a particle is given by $s = t^3 - 9t^2 + 24t - 18$ 1) Find the velocity when the acceleration is zero. 2) Find the acceleration when the velocity is zero.

Solution:

$$s = t^3 - 9t^2 + 24t - 18$$

$$v = \frac{ds}{dt} = 3t^2 - 18t + 24$$

$$a = \frac{d^2s}{dt^2} = 6t - 18$$

(i) Find velocity when acceleration is zero

i.e., $\frac{d^2s}{dt^2} = 0$

$\therefore 6t - 18 = 0$, $6t = 18$ $\therefore t = 3$ sec

Velocity $= 3\,(3)^2 - 18(3) + 24$

$= 27 - 54 + 24$

$= 51 - 54 = -3$ m/sec

(ii) Find the acceleration when the velocity is zero.

$\therefore\ v = 0$

$3t^2 - 18t + 24 = 0$

$t^2 - 6t + 8 = 0$

$(t - 2)\,(t - 4) = 0$

$\therefore\ t = 2$ or $t = 4$.

At $t = 2$ sec,

Acceleration $= 6(2) - 18$

$= 12 - 18$

$= -6$ m/sec^2

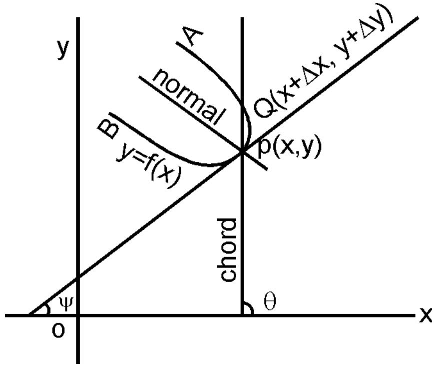
At $t = 4$ secs,

Acceleration $= 6(4) - 18$

$= 24 - 18$

$= 6$ m/ sec^2

4.3 TANGENTS AND NORMALS:



Let the graph of the function y=f(x) be represented by the curve AB,

Let P(x,y) be any point on the curve let $Q(x+\Delta x, y+\Delta y)$ be neighbouring point to P on the curve. Let the chord PQ makes an angle 'θ' with x axis. Let the tangent at P makes an angle ψ with X axis.

$$\text{Slope of the Chord } PQ = \frac{y+\Delta y - y}{x+\Delta x - x} = \frac{\Delta y}{\Delta x}$$

Now let the point Q moves along the curve towards the point P and coincide with P, so that $\Delta x \to 0$ and $\Delta y \to 0$ and in the limiting position PQ become the tangent at P.

$\therefore$ Slope of the tangent at P=slope of the chord when $Q \to P$

$$\therefore \tan\psi = \underset{Q \to P}{\text{Lt}} \tan\theta = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$\therefore$ The geometrical meaning of $\frac{dy}{dx}$ is the Slope of the tangent at a point (x,y) on it and it is denoted by 'm'

Note:

1. Equation of the line passing through (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$

2. If the slope of the tangent is 'm' the slope of the normal is $-\frac{1}{m}$

3. Slope of the line parallel to x axis is zero

4. Slope of the line parallel to y axis is ∞

4.3 WORKED EXAMPLES

PART - A

1.) Find the slope of the tangent at the point (1,1) on the curve $x^2 + 2y^2 = 3$

$$x^2 + 2y^2 = 3$$

Differentiate both sides w.r.t 'x'

$$2x + 4y\frac{dy}{dx} = 0$$

$$4y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{4y} = -\frac{x}{2y}$$

$$\text{At } x = 1, \ \frac{dy}{dx} = -\frac{1}{2}$$

$\therefore$ slope of the tangent at (1,1) is $-\frac{1}{2}$

2.) Find the slope of the tangent at the point x=2 to the curve $y = 5 - 2x - 3x^2$

Solution:

$$y = 5 - 2x - 3x^2$$

$$\frac{dy}{dx} = -2 - 6x$$

$$\text{At } x = 2, \ \frac{dy}{dx} = -2 - 12 = -14$$

$\therefore$ Slope of the tangent at x=2 is -14

3.) Find the slope of the normal at (1,-1) on the curve $y = 3x^2 - 4x$

Solution:

$$y = 3x^2 - 4x$$

$$\frac{dy}{dx} = 6x - 4$$

At $(1,-1)$ $\frac{dy}{dx} = 6 - 4 = 2$

$\therefore$ Slope of the tangent = 2

$\therefore$ Slope of the normal $= -\frac{1}{2}$

4.) Find the gradient of the curve $y^2 = 4x$ at (1,2)

Solution:

$$y^2 = 4x$$

$$2y\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

At (1,2) $\frac{dy}{dx} = \frac{2}{2} = 1$

Gradient of the curve = 1

5.) Find the point on the curve $y = 2x - x^2$ at which the tangent is parallel to x axis

Solution:

$$y = 2x - x^2$$

$$\frac{dy}{dx} = 2 - 2x$$

Since the tangent is parallel to x axis $\frac{dy}{dx}$ =0

$\therefore$2-2x=0   2x=2   x=1

When x=1, y=2-1=1

$\therefore$ The point is (1,1)

6.) Find the slope of the tangent at the point $\left(at^2, 2at\right)$ on the parabola $y^2 = 4ax$

Solution:

$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

At $\left(at^2, 2at\right)$ $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

$\therefore$ The slope of the tangent is $\frac{1}{t}$

PART - B

1.) Find the equation of the tangent and normal to the curve $y = 6 + x - x^2$ at (2,4).

Solution:

$$y = 6 + x - x^2$$

$$\frac{dy}{dx} = 1 - 2x$$

At (2,4) $\frac{dy}{dx} = 1 - 4 = -3$

$\therefore$ Slope of the tangent =-3

Equation of the tangent at (2,4) is $y - y_1 = m\left(x - x_1\right)$

$$y - 4 = -3\left(x - 2\right)$$

$$y - 4 = -3x + 6$$

$$3x + y - 10 = 0$$

slope of the normal = $\frac{1}{3}$

Equation of the normal is $y - y_1 = m\left(x - x_1\right)$

$$y-4=\frac{1}{3}(x-2)$$

$$3y-12=x-2$$

$$x-3y+10=0$$

2.) Find the equation of the tangent and normal to the curve $y=\frac{x+3}{x^2+1}$ at (2,1)

Solution:

$$y=\frac{x+3}{x^2+1}$$

$$\frac{dy}{dx}=\frac{(x^2+1)(1)-(x+3)2x}{(x^2+1)^2}=\frac{x^2+1-2x^2-6x}{(x^2+1)^2}$$

$$=\frac{1-6x-x^2}{(x^2+1)^2}$$

At (2,1) $\frac{dy}{dx}=\frac{1-12-4}{25}=\frac{-15}{25}=\frac{-3}{5}$

Equation of the tangent is $y-y_1=m(x-x_1)$

$$y-1=-\frac{3}{5}(x-2)$$

$$5(y-1)=-3(x-2)$$

$$5y-5=-3x+6$$

$$3x+5y-11=0$$

Slope of the normal $=\frac{5}{3}$

Equation of the normal is $y-y_1=m(x-x_1)$

$$y-1=\frac{5}{3}(x-2)$$

$$3y-3=5x-10$$

$$5x-3y-7=0$$

3.) Find the equation of the tangent to the curve $y = 3x^2 + 2x + 5$ at the point where the curve cuts the y-axis.

Solution:

The curve cuts y-axis $\therefore$ x=0

$$y = 3x^2 + 2x + 5$$

$$y = 0 + 0 + 5 = 5$$

The point is (0,5)

$$y = 3x^2 + 2x + 5$$

$$\frac{dy}{dx} = 6x + 2$$

At (0,5) $\quad \frac{dy}{dx} = 0 + 2 = 2$

Slope of the tangent =2

Equation of the tangent is $y - y_1 = m(x - x_1)$

$$y - 5 = 2(x - 0)$$

$$y - 5 = 2x$$

$$2x - y + 5 = 0$$

4) Find the equation of the tangent to the curve $y = (x-3)(x-4)$ at the point where the curve cuts the x-axis.

Solution:

If the curve cuts the x axis then y=0

$$0 = (x-3)(x-4)$$

$$\therefore x = 3 \text{ and } 4$$

$\therefore$ The points are (3,0) and (4.0)

$$y = (x-3)(x-4)$$

$$\frac{dy}{dx} = (x-3) + (x-4) = 2x - 7$$

At (3,0) $\quad \frac{dy}{dx} = 6 - 7 = -1$

Slope of the tangent at (3,0) is -1 Equation of the tangent at (3,0)

$$y - 0 = -1(x - 3)$$

is $\quad y = -x + 3$

$$x + y - 3 = 0$$

At (4,0) $\quad \frac{dy}{dx} = 8 - 7 = 1$

Slope of the tangent at (4,0) is 1.

Equation of the tangent at (4,0) is

$$y - 0 = 1(x - 4)$$

$$y = x - 4$$

$$x - y - 4 = 0$$

5.) Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$

Solution:

$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

At $(at^2, 2at)$ $\quad \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Is slope of the tangent $= \frac{1}{t}$

Equation of the tangent is $y - y_1 = m(x - x_1)$

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$yt - 2at^2 = x - at^2$$

$$x - yt + 2at^2 - at^2 = 0$$

$$x - yt + at^2 = 0$$

Slope of the normal =-t

Equation of the normal is

$$y - y_1 = m(x - x_1)$$
$$y - 2at = -t(x - at^2)$$
$$y - 2at = -xt + at^3$$
$$xt + y - 2at - at^3 = 0$$

6.) Find the equation of the tangent to the parabola $y^2 = 4ax$ which makes an angle 45° with X - aixs.

Solution:

$y^2 = 4ax$ 1

$$2y\frac{dy}{dx} = 4a$$

$$\text{slope of the tangent} = \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

Tangent makes an angle 45° with X - axis.

$\therefore$ Slope of the tangent = tan45° = 1

$$\therefore \frac{2a}{y} = 1$$

Put y=2a in (1)

$$4a^2 = 4ax$$
$$x = a$$

$\therefore$ The point is (a,2a)

Slope =1

$\therefore$ Equation of the tangent is $y - y_1 = m(x-x_1)$

$$y - 2a = 1(x - a)$$
$$y - 2a = x - a$$
$$x - y + a = 0$$

7.) Find the Equation of the tangent to the curve $y = 3x^2 - 6x + 1$ at the point where the tangent is parallel to the line $6x - y + 2 = 0$.

Solution:

$y = 3x^2 - 6x + 1$ …..(1)

$$\frac{dy}{dx} = 6x - 6$$

Slope of the given line = $-\frac{a}{b} = -\frac{6}{-1} = 6$

If the tangent is parallel to the given line then

$6x - 6 = 6$

$6x = 12 \qquad x = 2$

Put x=2 in (1)

$y = 3(2)^2 - 6(2) + 1$

$= 12 - 12 + 1 = 1$

$\therefore$ The point is (2,1)

Equation of the tangent is $y - y_1 = m(x - x_1)$

$y - 1 = 6(x - 2)$

$y - 1 = 6x - 12$

$6x - y - 11 = 0$

EXERCISE

PART - A

1.) If $A=x^2$, x=2 cm., $\frac{dx}{dt} = 0.6$ cm Find $\frac{dA}{dt}$

2.) Find $\frac{dA}{dt}$ if r=3 c.m $\frac{dr}{dt}$=0.2 c m and $A=\pi r^2$

3.) If $S=4\pi r^2$, r=1 cm , $\frac{ds}{dt}=4\pi$ c.m Find $\frac{dr}{dt}$

4.) If $V=\frac{4}{3}\pi r^3$, r=4c.m $\frac{dr}{dt}$=0.1 c.m Find $\frac{dV}{dt}$

PART - B

1.) The area of a square is increasing at the rate of 4 cm^2/sec. How fast its side is increasing when it is 8cm long?

2.) The radius of a circular plate is increasing at the rate of 0.03 cm/sec. At what rate the area is increasing when the radius 10cm.

3.) The circular path of oil spreads on water. The area is increasing at the rate of 16 cm^2/sec. How fast is the radius increasing when the radius is 8cm.

4.) The distance 's' meters travelled by a body in 't' sec is given by $s=80t-16t^2$. Find the velocity and acceleration after 2 seconds.

5.) The distance time formula of a moving particle is given by $s=2\cos 3t + 3\sin 3t$. Prove that the acceleration varies as its distance.

6.) Find the slope of the tangent to the curve xy=16 at the point (2,8).

7.) Find the slope of the normal to the curve $y=3x^2-4x$ at (1,-1).

8.) The radius of a sphere is expanding uniformly at the rate of 2cm/sec. Find the rate at which its surface area is increasing when the radius is 10cm.

9.) The radius of a spherical balloon is increasing at the rate of 5cm/sec. Find the rate of increasing in the volume of the balloon, when its radius is 30cm.

10.) A stone thrown in to still water causes a series of concentric ripples. If the radius of outer ripple is increasing at the rate of 5m/sec how fast is the area increasing when the outer ripple has a radius of 12m

11.) An inverted cone has a depth of 10cm and a base of radius 5cm. Water is poured into it at the rate of 1.5 c.c/sec. Find the rate at which the level of the water in the cone is rising when the depth is 4.cm.

12.) A. man 2m tall, walks directly away from a light 5m above the ground at the rate of $\frac{2}{3}$ m/sec. Find the rate at which his shadow lengthens.

13.) The distance travelled by a particle in time 't' seconds is $s = t^3 - 6t^2 + 12 + 8$. Find the velocity when the acceleration vanishes. Find also the acceleration when the velocity vanishes.

14.) The distance time formula of a moving particle is given by $s = 2t^3 + 3t^2 - 72t + 1$. Find the acceleration when the velocity vanishes. Find also the initial velocity.

15.) A Particle is moving in a st. line. Its distance at time 't' is given by $s = 2t^3 - 15t^2 + 36t - 70$

a) Find the initial velocity

b) Find the time when the velocity is zero

c) Find the time when the acceleration is zero

16.) The Velocity 'v' of a particle moving along a straight line when at a distance 's' from the origin is given by $s^2=m+nv^2$. Show that the acceleration of the particle is $\frac{s}{n}$

17.) A body moves in a straight line in such a manner that $s = \frac{1}{2}vt$'s' being the space travelled in time 't' and 'v' is the velocity at the end of time 't' prove that the acceleration is a constant.

18.) Find the equation of the tangent and normal to the following curves.

a.) $y = 2 - 3x + 4x^2$ at $x = 1$

b.) $y = 6 - x + x^2$ at $(2,8)$

c.) $y = 6 + x - x^2$ at $(2,4)$

d.) $y = \frac{6x}{x^2 - 1}$ at $(-2,-4)$

e.) $y = \frac{x+3}{x^2+1}$ at $(2,1)$

f.) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\,\text{Cos}\theta, b\sin\theta)$

g.) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$

19.) Find the equation of the tangents at the points where the curve $y = x^2 - x - 2$ meets the x-axis.

20.) Find the equation of the tangent to the curve $y = x^2 + 5x + 7$ where it cuts the y axis.

ANSWERS

PART - A

1.) $2.4 cm^2$ 2.) $1.2\pi cm^2$ 3.) $\frac{1}{2} cm$ 4.) $6.4\pi cm^3$

PART - B

1.) 0.25 2.) $0.6\pi cm^2 / sec$ 3.) $\frac{1}{\pi}$

4.) 16m/s,-32m/sec^2 6.) -4 7.) $-\frac{1}{2}$ 8.)160π

9.) 18000π cm^3/sec 10.) 120π m^2/sec

11.) $\frac{1.5}{4\pi}$ cm/sec 12.) $\frac{4}{9}$ m/sec 13.) 0

14.) 42,-72 15.) (a) 36 (b) t=2 or 3 (c) $5/2$

18.) a.) $5x - y - 2 = 0, \ x + 5y - 16 = 0$

b.) $3x - y + 2 = 0, \ x + 3y - 26 = 0$

c.) $10x + 3y + 32 = 0, 3x - 10y - 34 = 0$

d.) $3x + 5y - 11 = 0, \ 5x - 3y - 7 = 0$

e.) $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \ \frac{ax}{\sec\theta} - \frac{by}{\tan\theta} = a^2 + b^2$

f.) $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1, \ \frac{ax}{\sec\theta} + \frac{by}{\sec\theta} = a^2 + b^2$

19.) $3x - y - 6 = 0$, $3x + y + 3 = 0$

20.) $5x - y + 7 = 0$

UNIT- V

APPLICATION OF DIFFERENTIATION – II

5.1 Definition of Increasing function, Decreasing function and turning points. Maxima and Minima (for single variable only) – Simple Problems.

5.2 Partial Differentiation

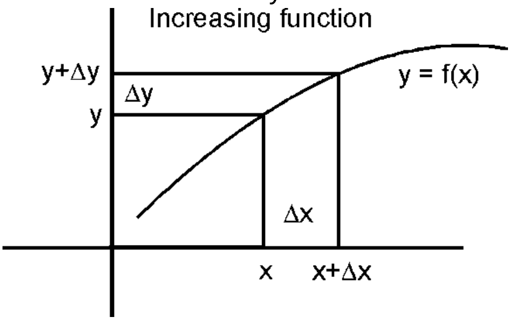
Partial differentiation of two variable up to second orders only. Simple problems.

5.3 Definition of Homogeneous functions Eulers theorem Simple Problems.

5.1 APPLICATION OF DIFFERENTIATION-II

Increasing function:

A function y = f(x) is said to be increasing function if value of y increases as x increases or value of y decreases as x decreases



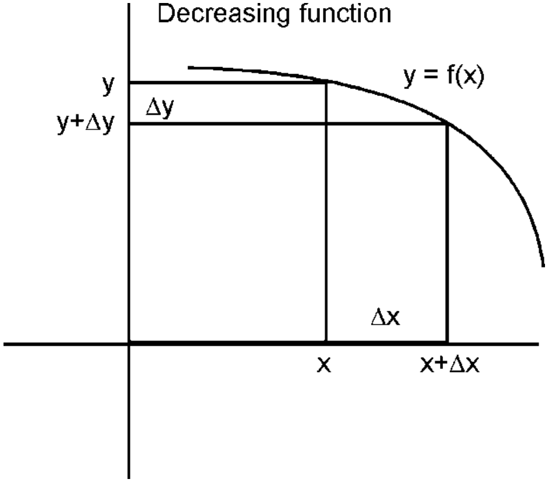
In the above figurey = f(x) is a increasing function and $\Delta x > 0, \Delta y > 0$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{+}{+} = +ve$$

$\therefore$ At every point on the increasing function the value of $\frac{dy}{dx}$ is positive

Decreasing function:

A function y=f(x) is said to be decreasing function if the value of y decreases as x increases or value of y increases as x decreases



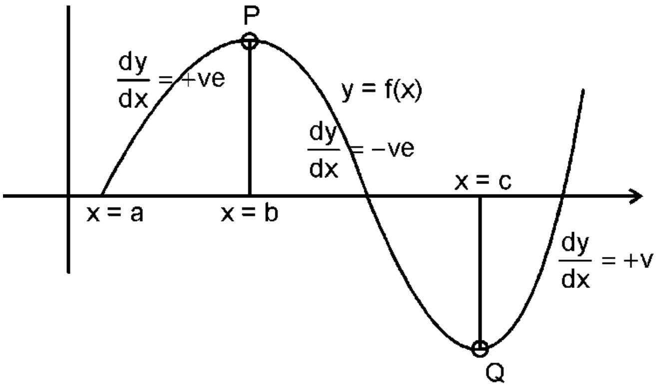
In the above figure y=f(x) is a decreasing function and $\Delta x > 0$ and $\Delta y < 0$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\Delta y}{\Delta x} = \frac{-}{+} = -ve$$

$\therefore$ At every point on the decreasing function the value of $\frac{dy}{dx}$ is negative

Turning points:

A function need not always be increasing or decreasing . In most cases the function is increasing in some interval and decreasing in some other interval.



In the above figure the function is increasing in the interval [a,b] and [c,∞]

It is decreasing in the interval[b,c]

Definition of turning points:

Turing point is the point at which the function changes either from decreasing to increasing or from increasing to decreasing.

In the above figure P and Q are turning points.

Since at the point P, the function changes from increasing to decreasing, valueof $\frac{dy}{dx}$ changesfrom positive to negative. Hence at the point P, $\frac{dy}{dx}=0$

Similarly at the point Q, $\frac{dy}{dx}=0$

Maximum of a function:

The maximum value of a function y =f(x) is the ordinate (y coordinate) of the turning point at which the function changes from increasing to decreasing function.

$\therefore$ At maximum, $\frac{dy}{dx}=0$

Minimum of a function:

The minimum value of a function y =f(x) is the ordinate (y coordinate) of the turning point at which the function changes from decreasing to increasing function.

$\therefore$ At minimum, $\frac{dy}{dx}=0$

Conditions for maximum:

At maximum, $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}$ is negative

Conditions for minimum:

At minimum, $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}$ is positive

5.1 WORKED EXAMPLES
PART-A

1.) Write the Condition for the function y=f(x) to be maximum

Solution: At x = a,

(i) $\frac{dy}{dx}=0$

(ii) $\frac{d^2y}{dx^2}<0$

2.) Write the condition for the function y=(x) to be minimum

Solution: At x = a,

(i) $\frac{dy}{dx}=0$

(ii) $\frac{d^2y}{dx^2}>0$

3.) For what value of x the function $y = x^2 - 4x$ will have maximum or minimum value

Solution:

$$y = x^2 - 4x$$

$$\frac{dy}{dx} = 2x - 4$$

$$\frac{dy}{dx} = 0 \Rightarrow 2x - 4 = 0$$

$$2x = 4 \quad \Rightarrow x = 2$$

At x =2, the function will have maximum or minimum value.

4.) For what value of x the function $y = x^2 - 10x$ will have maximum or minimum value

Solution: Let $y = x^2 - 10x$

$$\frac{dy}{dx} = 2x - 10$$

$$\frac{dy}{dx} = 0 \Rightarrow \quad 2x - 10 = 0$$

$$2x = 10 \quad x = 5$$

At x = 5, the function will have maximum or minimum value.

5.) For what value of x the function $y = \sin x$ will have maximum or minimum value

Solution:

$$y = \sin x, \frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = 0 \Rightarrow \cos x = \cos 90^\circ$$

$$x = 90^\circ \text{ or } \pi/2$$

At $x = \pi/2$, the function will have maximum or minimum value.

6.) Find the minimum value of $y = x^2 - 4x$

Solution:

$$y = x^2 - 4x$$

$$\frac{dy}{dx} = 2x - 4$$

$$\text{Put } \frac{dy}{dx} = 0$$

$$2x - 4 = 0$$

$$2x = 4$$

$$x = 2$$

$$\text{When } x=2,\ \frac{d^2y}{dx^2} = 2 > 0$$

$\therefore$ at $x = 2$ the function is minimum

Put x = 2 in y

$$y = x^2 - 4x = (2)^2 - 4(2) = 4 - 8 = -4$$

$\therefore$ y=-4 is the minimum value.

7.) Find the maximum value of $y = x - x^2$

Solution:

$$y = x - x^2$$

$$\frac{dy}{dx} = 1 - 2x$$

$$\text{Put } \frac{dy}{dx} = 0$$

$$\therefore 1 - 2x = 0$$

$$2x = 1$$

$$x = 1/2$$

$$\text{When } x = \frac{1}{2}, \frac{d^2y}{dx^2} = -2$$

when $x = \frac{1}{2}$ the function is maximum

Put $x = \frac{1}{2}$ in y,

$$y = x - x^2$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$\therefore\ y = \frac{1}{4}$ is the maximum value.

8.) Find the maximum or minimum value of $y = 4x - 2x^2$

Solution:

$$y = 4x - 2x^2$$

$$\frac{dy}{dx} = 4 - 4x$$

$$\frac{d^2y}{dx^2} = -4$$

Put $\frac{dy}{dx} = 0$

$$4 - 4x = 0$$

$$4x = 4$$

$$x = 1$$

when $x = 1, \frac{d^2y}{dx^2} = -4$

At x = 1, the function is maximum.

When $x = 1, y = 4x - 2x^2$

$= 4(1) - 2(1)^2 = 2$ is the maximum value

9.) Find the maximum or minimum value of $y = 2x - 3x^2$

Solution:

$$y = 2x - 3x^2$$

$$\frac{dy}{dx} = 2 - 6x$$

$$\frac{d^2y}{dx^2} = -6$$

$$\text{Put } \frac{dy}{dx} = 0$$

$$2 - 6x = 0$$

$$2 = 6x$$

$$x = \frac{1}{3}$$

$$x = \frac{1}{3}, \frac{d^2y}{dx^2} = -6$$

When $x = 1/3$, the function is maximum.

$$\text{Put } x = \frac{1}{3} \text{ in y,}$$

$$y = 4x - 2x^2$$

$$= 4\frac{1}{3} - 2\left(\frac{1}{3}\right)^2$$

$$= \frac{4}{3} - 2\frac{1}{9} = \frac{12-2}{9} = \frac{10}{9} \text{ is the maximum value.}$$

PART - B

1.) Find the maximum and minimum values of the function

$$2x^3 - 3x^2 - 36x + 10$$

Solution:

$$y = 2x^3 - 3x^2 - 36x + 10$$

$$\frac{dy}{dx} = 6x^2 - 6x - 36$$

$$\frac{d^2y}{dx^2} = 12x - 6$$

$$\text{Put } \frac{dy}{dx} = 0$$

$$6x^2 - 6x - 36 = 0$$

$$\text{i.e., } x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x-3=0 \quad \Big| \quad x+2=0$$
$$x=3 \quad \Big| \quad x=-2$$

When $x=3, \dfrac{d^2y}{dx^2}=12x-6=12\times 3-6=36-6=30$

When x=3, the function is minimum

Put $x=3$ in y,

$$y=2(3)^3-3(3)^2-36\times 3+10$$
$$=54-27-108+10=-71$$

$\therefore\ y=-71$ is the minimum value

When $x=-2, \dfrac{d^2y}{dx^2}=12(-2)-6=-30$

When $x=-2$, the function is maximum

Put $x=-2$ in y,

$$y=2x^3-3x^2-36x+10$$
$$2(-2)^3-3(-2)^2-36(-2)+10$$
$$=-16-12+82=54$$

$\therefore\ y=54$ is the minimum value

2.) Find the maximum and minimum values of $(x-1)^2(x-2)$

Solution:

Let $y=(x-1)^2(x-2)$

$$\frac{dy}{dx}=(x-1)^2(1)+(x-2)2(x-1)(1)$$
$$=(x-1)^2+2(x-1)(x-2)$$
$$=(x-1)[x-1+2(x-2)]$$
$$\frac{dy}{dx}=(x-1)(3x-5)$$
$$\frac{d^2y}{dx}=(x-1)(3)+(3x-5)(1)$$
$$=3x-3+3x-5$$
$$=6x-8$$

Put $\frac{dy}{dx} = 0$

$(x-1)(3x-5) = 0$

$$\begin{array}{c|c} x-1=0 & 3x-5=0 \\ x=1 & x=\frac{5}{3} \end{array}$$

(i) When $x = 1, \frac{d^2y}{dx^2} = 6(1) - 8 = -2$

when $x = 1$ the function is maximum

Put $x = 1$ in y,

$y = (1-1)^2(1-2) = 0$

$\therefore$ $y = 0$ is the maximum value

(ii) When $x = \frac{5}{3}, \frac{d^2y}{dx^2} = 6\left(\frac{5}{3} - 8\right) = 10 - 8 = 2$

when $x = \frac{5}{3}$ the function is minimum

Put $x = \frac{5}{3}$ in y,

$$y = \left(\frac{5}{3} - 1\right)^2 \left(\frac{5}{3} - 2\right)$$

$$= \left(\frac{2}{3}\right)^2 . \left(-1/3\right)$$

$$= \frac{4}{9} \times \frac{-1}{3}$$

$$= \frac{-4}{27}$$

$\therefore$ $y = \frac{-4}{27}$ is the minimum value

3.) Find the maximum value of $\frac{\log x}{x}$ for positivevalue of x.

Solution:

Let $y = \frac{\log x}{x}$

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \log x.(1)}{x^2} = \frac{1 - \log x}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2(\frac{-1}{x}) - (1 - \log x)(2x)}{x^4}$$

$$= \frac{-x - 2x + 2x.\log x}{x^4} = \frac{2\log x - 3}{x^3}$$

Put $\frac{dy}{dx} = 0$

$$\frac{1 - \log x}{x^2} = 0$$

$$1 - \log x = 0$$

$$\log x = 1$$

$$\log_e x = 1$$

$$x = e^1 = e$$

$$\therefore x = e$$

When $x = e, \frac{d^2y}{dx^2} = \frac{2\log e - 3}{e^3}$

$$= \frac{2 - 3}{3} = \frac{-1}{3}$$

When $x = e$ the function is maximum

Put $x = e$ in y,

$$y = \frac{\log x}{x} = \frac{\log e}{e} = \frac{1}{e}$$

$\therefore\ y = \frac{1}{e}$ is the maximum value

4.) Find the minimum value of xlogx.

Solution:

Let $y = x\log x$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x.(1) = 1 + \log x$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}$$

Put $\frac{dy}{dx} = 0$

$$1 + \log x = 0$$

$$\log x = -1;\ \log_e x = -1$$

$$x = e^{-1}$$

$$= x = \frac{1}{e}$$

When $x = \frac{1}{e}, \frac{d^2y}{dx^2} = \frac{1}{x} = \frac{1}{\frac{1}{e}} = e$

At $x = \frac{1}{e}$, the function is minimum

Put $x = \frac{1}{e}$ in y,

$$y = x.\log x = \frac{1}{e}\log\left(\frac{1}{e}\right)$$

$$y = \frac{1}{e}[\log 1 - \log e]$$

$$= \frac{1}{e}[0 - \log e] = \frac{1}{e}[-1] = \frac{-1}{e}$$

$\therefore\ y = \frac{-1}{e}$ is the minimum value

5.) Show that the function $f(x) = \sin x(1 + \cos x)$ is maximum at $x = \frac{\pi}{3}$

Solution: Lety= sinx(1+cosx)

$$\frac{dy}{dx} = \sin x(-\sin x) + (1 + \cos x)\cos x$$

$$= -\sin^2 x + \cos x + \cos^2 x$$

$$= \cos^2 x - \sin^2 x + \cos x$$

$$= \cos 2x + \cos x$$

$$\frac{d^2y}{dx^2} = -2\sin 2x - \sin x$$

Put $\frac{dy}{dx} = 0$

$$\cos 2x + \cos x = 0$$

$$2\cos\frac{3x}{2}.\cos\frac{x}{2} = 0 \qquad (\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2})$$

$\cos\frac{3x}{2} = \cos\frac{\pi}{2}$	$\cos\frac{x}{2} = \cos\frac{\pi}{2}$
$\frac{3x}{2} = \frac{\pi}{2}$	$\frac{x}{2} = \pi/2$
$x = \frac{\pi}{3}$	$x = \pi$

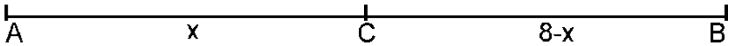
$$\text{When} x = \frac{\pi}{3}, \frac{d^2y}{dx^2} = -4\text{Sin}\frac{\pi}{3}\text{Cos}\frac{\pi}{3} - \text{Sin}\frac{\pi}{3}$$

$$= -4\left[\frac{\sqrt{3}}{2}\right]\left[\frac{1}{2}\right] - \frac{\sqrt{3}}{2}$$

$$= \sqrt{3} - \frac{\sqrt{3}}{2} = -3\frac{\sqrt{3}}{2}$$

The function is maximum at $x = \frac{\pi}{3}$

6.) AB is 8cm long, find a point C on AB so that $3AC^2 + BC^2$ may be a minimum



Solution:

Let: AC = x so that BC=8-x

i.e $y = 3AC^2 + BC^2$

$= 3x^2 + (8 - x)^2$

$= 3x^2 + 16 - 16x + x^2$

$= 4x^2 - 16x + 16$

$\frac{dy}{dx} = 8x - 16$

$\frac{d^2y}{dx^2} = 8$

Put $\frac{dy}{dx} = 0$

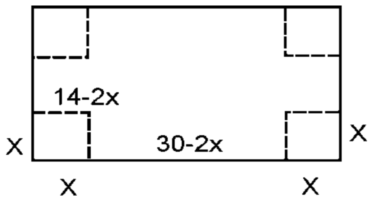
$8x - 16 = 0 \Rightarrow \quad 8x = 16 \Rightarrow x = 2$

when $x = 2, \frac{d^2y}{dx^2} = 8$

When x = 2 the function y is the minimum

$\therefore$ The point C is at 2cm fromA

7.) A rectangle sheet of metal is 30 cm by 14 cm. Equal squares are cut off at the four corners and the remainder is folded up to form an open rectangular box. Find the side of the square out off theso that the volume of the box may be maximum.



Solution:

Let the side of the square be x

Length of the box =30-2x

Breadth of the box=14-2x

Height of the box=x

$\therefore$Volume of the box = V=(30-2x)(14-2x)(x)

$$V = \left(420 - 60x - 28x + 4x^2\right)x \qquad [\because \text{volume} = l \times b \times h]$$

$$= x\left(4x^2 - 88x + 420\right)$$

$$\therefore V = 4x^3 - 88x^2 + 420x$$

$$\frac{dV}{dx} = 12x^2 - 176x + 420$$

$$\frac{d^2V}{dx^2} = 24x - 176$$

Put $\frac{dV}{dx} = 0$

$$12x^2 - 176x + 420 = 0$$

i.e.,

$$3x^2 - 44x + 105 = 0$$

$$3x^2 - 9x - 35x + 105 = 0$$

$$3x(x-3) - 35(x-3) = 0$$

$$x - 3 = 0 \quad\Big|\quad 3x - 35 = 0$$

$$x = 3 \quad\Big|\quad x = \frac{35}{3}$$

when $x = 3, \frac{d^2V}{dx^2} = 24(3) - 176$

$$= 72 - 176 = -104$$

At x = 3the volume is maximum

Hence the volume is maximum when the side of the square cut from each of the corner is 3 cm.

8.) Find two numbers whose sum is 10 and whose product is maximum.

Solution: let the numbers be x and y

Given, sum = 10

i.e., x + y =10, y=10-x

Let p be the product of the numbers

$$p = xy = x(10 - x) = 10x - x^2$$

$$\frac{dp}{dx} = 10 - 2x, \frac{d^2p}{dx^2} = -2$$

Put $\frac{dp}{dx} = 0, 10 - 2x = 0, x = 5$

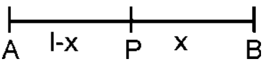
$$\text{At } x = 5, \left(\frac{d^2p}{dx^2}\right) = -2$$

$\therefore$ P is maximum when x=5

$\therefore$ The Numbers are 5,5

9.) The bending moment at B at a distance x from one end of beam of length 'L' uniformly loaded is given by $M = \frac{1}{2}wL\,x - \frac{1}{2}wx^2$ where x=load per unit length. Show that the maximum bending moment is at the centre on the beam.

Solution:



Bending moment $M = \frac{1}{2}wL\,x - \frac{1}{2}wx^2$

$$\frac{dM}{dx} = \frac{1}{2}wL - wx$$

$$\frac{d^2M}{dx^2} = -w$$

Put $\frac{dM}{dx} = 0$

$$\therefore \frac{1}{2}wL - wx = 0$$

$$wx = \frac{1}{2}wL$$

$$x = \frac{L}{2}$$

When $x = \frac{L}{2}, \ \frac{d^2M}{dx^2} = -w$

$\therefore$ at $x = \frac{L}{2}$, M is maximum.

$\therefore$ The maximum bending moment is at the centre of the beam.

10.) A body moves so that its distance from a given point at time 't' seconds, is given by $S = 4t + \frac{3}{2}t^2 - \frac{2}{3}t^3$ Find when the body attains its maximum velocity.

Solution:

$$S = 4t + \frac{3}{2}t^2 - \frac{2}{3}t^3$$

Velocity $v = \frac{ds}{dt} = 4 + \frac{3}{2}.2t - \frac{2}{3}.3t^2$

velocity $v = \frac{ds}{dt} = 4 + 3t - 2t^2$

$$\frac{dv}{dt} = 3 - 4t$$

$$\therefore \frac{d^2v}{dt^2} = -4$$

put $\frac{dv}{dt} = 0$

$$3 - 4t = 0$$

$$t = \frac{3}{4}$$

When $t = \frac{3}{4}, \frac{d^2v}{dt^2} = -4$

At t = $\frac{3}{4}$ seconds v is maximum

$\therefore$ The body attains its maximum velocity when $t = \frac{3}{4}$ sec.

5.2. PARTIAL DIFFERENTIATION

Functions of two or more variables

In many applications, we come across function involving more than one independent variable. For example, the area of rectangle is a function of two variables, the length and breadth of the rectangle.

Definition: Let u= f(x, y) be a function of two independent variables x and y. The Derivative of u with respect to x when x varies and y remains constant is called the partial derivative of u with respect to x and is denoted by $\frac{\partial u}{\partial x}$

$$\therefore \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$\frac{\partial u}{\partial x}$ = is also written as $\frac{\partial f(x,y)}{\partial x}$ or $\frac{\partial f}{\partial x}$

Similarly when x remains constant and y varies, the derivative of u with respect to y is called the partial derivative of u with respect to y and is denoted by $\frac{\partial u}{\partial y}$.

$$\therefore \frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$\frac{\partial u}{\partial y}$ is also written as $\frac{\partial f(x,y)}{\partial y}$ or $\frac{\partial f}{\partial y}$

Second order Partial derivatives

In General, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are also function of x and y. They can be further differentiated partially w.r.t.x and y as follows.

Hence (i) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \text{(or)} \frac{\partial^2 f}{\partial x^2} \text{(or)} f_{xx}$

(ii) $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial x\partial y}(\text{or})\frac{\partial^2 f}{\partial x\,\partial y}(\text{or})f_{xy}$$

(iii) $$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial y\partial x}(\text{or})\frac{\partial^2 f}{\partial y\,\partial x}(\text{or})f_{yx}$$

(iv) $$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial y^2}(\text{or})\frac{\partial^2 f}{\partial y^2}(\text{or})f_{yy}$$

Generally, for all ordinary functions,

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\,\partial x}$$

5.2 WORKED EXAMPLES
PART –A

1.) If $u = x^4 + y^3 + 3x^2.y^2 + 3x^2y$, Find $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2},\frac{\partial^2 u}{\partial x\partial y}$,and $\frac{\partial^2 u}{\partial y\partial x}$

a.) $$\frac{\partial u}{\partial x}=4x^3+6xy^2+6xy$$

b.) $$\frac{\partial u}{\partial y}=3y^3+6x^2y+3x^2$$

c.) $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(4x^3+6xy^2+6xy\right)=12x^2+6y^2+6y$$

d.) $$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(3y^2+6x^2y+3x^2\right)=6y+6x^2$$

e.) $$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(3y^2+6x^2y+3x^2\right)=12xy+6x$$

f.) $$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(4x^3+6xy^2+6xy\right)=12xy+6x$$

2.) If u =x. sin y + y. sin x, find all second order partial derivatives .

Solution: u =x. sin y + y. sin x,

a.) $\frac{\partial u}{\partial x} = \sin y.\frac{\partial(x)}{\partial x} + y.\frac{\partial^{(\sin x)}}{\partial x}$

$= \sin y.1 + y\,\text{Cos}x$

b.) $\frac{\partial u}{\partial y} = x.\frac{\partial(\sin y)}{\partial y} + \sin x.\frac{\partial(y)}{\partial y}$

$= x.\cos y + \sin x.1$

c.) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x.\cos y + \sin x)$

$= \cos y + \cos x$

d.) $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(\sin y + y\cos x)$

$= \cos y + \cos x$

e.) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(\sin y + y\cos x)$

$= -y.\sin x$

f.) $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(x\cos y + \sin x)$

$= -x.\sin y$

3.) If $u = x^3 + y^3 + 3xy$, find (i) $\frac{\partial u}{\partial x}$ (ii.) $.\frac{\partial u}{\partial y}$ (iii.) $.\frac{\partial^2 u}{\partial x^2}$

(iv) $\frac{\partial^2 u}{\partial y^2}$ (v) $\frac{\partial^2 u}{\partial x \partial y}$ (vi) $\frac{\partial^2 u}{\partial y.\partial x}$

Solution:

$u = x^3 + y^3 + 3xy,$

(i) $\frac{\partial u}{\partial x} = 3x^2 + 3y$

(ii) $\frac{\partial u}{\partial y} = 3y^2 + 3x$

(iii) $\frac{\partial^2 u}{\partial x^2} = 6x$

(iv) $\frac{\partial^2 u}{\partial y^2} = 6y$

(v) $\frac{\partial^2 u}{\partial x \partial y} = 3$ (vi) $\frac{\partial^2 u}{\partial y \partial x} = 3$

4.) Find $\frac{\partial u}{\partial x}$ When $u = \log\left(x^2 + y^2\right)$

$$u = \log\left(x^2 + y^2\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}.2x = \frac{2x}{x^2 + y^2}$$

5.). Find $\frac{\partial u}{\partial x}$ When $u = \sin^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}.\frac{1}{y}.1 = \frac{y}{\sqrt{y^2 - x^2}}\frac{1}{y} = \frac{1}{\sqrt{y^2 - x^2}}$$

6.) If u =sin 3x. cos 4 y Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

If u =sin 3xcos 4 y

$\frac{\partial u}{\partial x}$ = cos 4 y cos 3x 3

$\frac{\partial u}{\partial y}$ = sin 3 x (– sin 4y.4) = -4sin3xsin4y

7.) If u= $e^{x^2+y^2}$ Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

u= $e^{x^2+y^2}$

$$\frac{\partial u}{\partial x} = e^{x^2+y^2} 2x \,, \frac{\partial u}{\partial y} = e^{x^2+y^2} 2y$$

8.) Find $\frac{\partial u}{\partial y}$ if u= $\tan^{-1}\left(\frac{y}{x}\right)$

Solution:u= $\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(\frac{1}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}.\frac{1}{x} = \frac{x}{x^2+y^2}$$

9.) If u= $x^3 + y^3 + 3x^2y + 3xy^2$, find $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial x} = 3x^2 + 3y.2x + 3y^2$$

$$= 3x^2 + 6xy + 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6y$$

$$\frac{\partial u}{\partial y} = 3y^2 + 3x^2 + 3x.2y$$

$$\frac{\partial^2 u}{\partial y^2} = 6y + 6x$$

Here $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

PART - B

1.) If $x^2 + 4xy + y^2$, Show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution: $u = x^2 + 4xy + y^2$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 4xy + y^2\right) = 2x + 4y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 4xy + y^2\right) = 4x + 2y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(4x + 2y) = 4 \qquad \text{...1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(2x + 4y) = 4 \qquad \text{...2}$$

2.) If u= $x^3 + y^3 + 3xy^2$, Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$

Solution: $u = x^3 + y^3 + 3xy^2$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + y^3 + 3xy^2\right) = 3x^2 + 3y^2$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + y^3 + 3xy^2\right) = 3y^2 + 6xy$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left(3x^2 + 3y^2\right) + y\left(3y^2 + 6xy\right)$$

$$= 3x^3 + 3xy^2 + 3y^3 + 6xy^2$$

$$= 3x^3 + 3y^3 + 9xy^2$$

$$= 3\left(x^3 + y^3 + 3xy^2\right) = 3u$$

5.3 HOMOGENEOUS FUNCTIONS:

An expression in which each term contains equal powers is called a homogeneous expression.

A function of severable variables is said to be homogeneous of degree 'n'if multiplying each variable by t(where t>0) has the same effect as multiplying the original function by t^n. Thus f(x,y) is homogeneous function of degree n if f(t x, t y)= t^n f(x, y).

Example:

If $u = \frac{1}{\sqrt{x^2 + y^2}}$ find the degree

Solution: Put x= t x, andy= t y

$$u = \frac{1}{\sqrt{(tx)^2 + (ty)^2}} = \frac{1}{\sqrt{t^2x^2 + t^2y^2}} = \frac{1}{t\sqrt{x^2 + y^2}}$$

$$= \frac{t^{-1}}{\sqrt{x^2 + y^2}}$$

u is a homogeneous function of degree -1.

Euler's theorem on homogeneous functions:

If u is a homogeneous function of x, y of degree n, then

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = n.u$$

Proof: If u is a homogeneous function of x,y

$$u = f(x,y) = x^n.\phi\left(\frac{y}{x}\right) \qquad 1$$

Differentiating (1) Partially wrt x'

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n\phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) - x^{n-2}y\phi'\left(\frac{y}{x}\right) \qquad 2$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\frac{1}{x}$$

$$= x^{n-1}\phi'\left(\frac{y}{x}\right) \qquad 3$$

Multiplying (2) by x and (3) by y and adding.

$$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) - x^{n-1}y\phi'\left(\frac{y}{x}\right) + x^{n-1}y\phi\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{by} \quad (1)$$

5.3 WORKED EXAMPLES
PART–A

1.) Find the degree of the function $u = \frac{x^3 - y^3}{x - y}$

Solution:

$$u = \frac{x^3 - y^3}{x - y}$$

Put x=tx,y=ty

$$u = \frac{t^3x^3 - t^3y^3}{tx - ty} = \frac{t^3\left(x^3 - y^3\right)}{t(x - y)}$$

$$= \frac{t^2\left(x^3 - y^3\right)}{x - y}$$

$\therefore$degree of the function is 2

2.) Find the degree of the function $u = x^3 + 9xy^2 + y^3$

Solution:

$$u = x^3 + 9xy^2 + y^3$$

Putx = t x, and y= t y

$$u = (tx)^3 + 9(tx).(ty)^2 + (ty)^3$$
$$= t^3x^3 + 9xt^3y^2 + t^3y^3$$
$$= t^3\left[x^3 + 9xy^2 + y^2\right]$$

$\therefore$ degree of the function is 3

PART-B

1.) Verify Euler's theorem when $u = x^3 - 2x^2y + 3xy^2 + y^3$

Solution:

$$u = x^3 - 2x^2y + 3xy^2 + y^3$$
$$u(tx, ty) = t^3x^3 - 2(tx)^2ty + 3(tx)(ty)^2 + (ty)^3$$
$$= t^3x^3 - 2t^2x^2ty + 3txt^2y^2 + t^3y^3$$
$$= t^3\left[x^3 - 2x^2y + 3xy^2 + y^3\right]$$

$\therefore$ Therefore u(x,y) is homogeneous function of degree 3.

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = nu$$

To prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{x} = 3u$$

$$\frac{\partial u}{\partial x} = 3x^2 - 4xy + 3y^2 + 0 = 3x^2 - 4xy + 3y^2$$

$$\frac{\partial u}{\partial y} = 0 - 2x^2 + 3x2y + 3y^2 = 6xy - 2x^2 + 3y^2$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3 - 4x^2y + 3xy^2 + 6xy^2 - 2x^2y + 3y^3$$
$$= 3\left[x^3 - 2xy^2y + 3xy^2 + y^3\right] = 3u$$

Hence Euler's theorem is verified.

2.) Verify Euler's theorem for $u(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$

$$u(x,y)=\frac{1}{\sqrt{x^2+y^2}}$$

$$u(tx,ty)=\frac{1}{\sqrt{(tx)^2+(ty)^2}}=\frac{1}{\sqrt{t^2x^2+t^2y^2}}=\frac{1}{t\sqrt{x^2+y^2}}$$

$$=t^{-1}\frac{1}{\sqrt{x^2+y^2}}$$

$\therefore$ u is a homogeneous function of order -1.

By Euler's theorem $x\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=(-1)u$

Verification: $u=\dfrac{1}{\sqrt{x^2+y^2}}=\left(x^2+y^2\right)^{-1/2}$

$$\frac{\partial u}{\partial x}=-1/2(x^2+y^2)^{-3/2}2x$$

$$=-x(x^2+y^2)^{-3/2}$$

$$\frac{\partial u}{\partial y}=\frac{-1}{2}(x^2+y^2)^{-3/2}2y$$

$$=-y(x^2+y^2)^{-3/2}$$

$$\therefore\ x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=x.\left[-x(x^2+y^2)^{-3/2}\right]+y\left[-y(x^2+y^2)^{-3/2}\right]$$

$$=-x^2(x^2+y^2)^{-3/2}-y^2(x^2+y^2)^{-3/2}$$

$$=-(x^2+y^2).(x^2+y^2)^{-3/2}$$

$$=-(x^2+y^2)^{1-\frac{3}{2}}=-(x^2+y^2)^{-\frac{1}{2}}=\frac{-1}{\sqrt{x^2+y^2}}=-u \qquad (2)$$

From (1) and (2) Euler's theorem is verified

3.) Using Euler's theorem, for $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

prove that $x\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \tan u.$

Solution:

U is not a homogeneous function. But sinu is a homogeneous function.

$$\text{Define } \sin u = \frac{x^2+y^2}{x+y}$$

$$\therefore f(x,y) = \sin u = \frac{x^2+y^2}{x+y}$$

$$f(tx,ty) = \frac{(tx)^2+(ty)^2}{tx+ty}$$

$$= \frac{t^2(x^2+y^2)}{t(x+y)}$$

$$= t\left[\frac{x^2+y^2}{x+y}\right]$$

$$= t\ f(x,y)$$

$\therefore$sinu is a homogeneous function of degree 1.

By Euler's theorem

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = 1\sin u$$

$$x\cos u.\frac{\partial u}{\partial x} + y\ \cos u\frac{\partial u}{\partial y} = 1\sin u$$

$$\cos u\left[x\ \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \sin u$$

$$\therefore\quad x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \tan u$$

4. Using Euler's theoremif $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$,

prove that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \sin 2u.$

Solution:

u is not a homogeneous function. But tanu is a homogeneous function.

$$\therefore \quad \tan u = \frac{x^3+y^3}{x-y}$$

$$\tan u = \frac{(tx)^3+(ty)^3}{tx-ty} = \frac{t^3x^3+t^3y^3}{tx-ty}$$

$$= \frac{t^3\left(x^3+y^3\right)}{t(x-y)} = t^2\,\frac{x^3+y^3}{x-y}$$

$\therefore$tanu is a homogeneous function of degree 2.

By Euler's theorem,

$$x.\frac{\partial}{\partial x}(\tan u) + y.\frac{\partial(\tan u)}{\partial y} = 2\tan u$$

$$x.\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\tan u$$

$$\div \sec^2 u, \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u\cos^2 u}{\cos u} = 2\sin u\cos u = \sin 2u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

5.) Using Euler's theorem Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

If $u = \sin^{-1}\left(\frac{x-y}{\sqrt{x}+\sqrt{y}}\right)$

Solution:

u is not homogeneous and hence take the function sinu

$$u = \sin^{-1}\left(\frac{x-y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\sin u = \frac{x-y}{\sqrt{x}+\sqrt{y}}$$

Put $x = xt, y = yt$

$$f(tx, ty) = \frac{xt-yt}{\sqrt{xt}+\sqrt{yt}}$$

$$= \frac{t(x-y)}{\sqrt{t}\left(\sqrt{x}+\sqrt{y}\right)} = t^{1/2}\frac{(x-y)}{\sqrt{x}+\sqrt{y}}$$

$\therefore$ Sinu is a homogeneous function of order $\frac{1}{2}$

Using Euler's theorem.

$$x.\frac{\partial(\sin u)}{\partial x} + y.\frac{\partial(\sin u)}{\partial x} = \frac{1}{2}\sin u$$

$$x(\cos u)\frac{\partial u}{\partial x} + y\text{Cos}u)\frac{\partial u}{\partial y} = 1/2 \sin u$$

$$\div \cos u, \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$$

6.) Using Euler's theorem if $u = xy^2 \sin\left(\frac{x}{y}\right)$ show that

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 3u$$

Solution:

$$u = xy^2 \sin\left(\frac{x}{y}\right)$$

Put x =tx, y =ty

$$u = tx(ty)^2 \sin\left(\frac{tx}{ty}\right)$$

$$u = t^3xy^2 \sin\left(\frac{x}{y}\right)$$

$\therefore$ u homogeneousfunction of order 3.

Using Euler's theorem, we have

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 3u$$

7.) If $z = e^{x^3+y^3}$ Prove that $x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = 3z\log z$

Solution:

$$z = e^{x^3+y^3}$$

$$\log z = \log e^{x^3+y^3}$$

$$= \left(x^3 + y^3\right)\log e$$

$$\log z = x^3 + y^3 = f(x,y)$$

$$f(tx,ty) = t^3.x^3 + t^3.y^3 = t^3\left(x^3 + Y^3\right)$$

$$= t^3(x,y) = t^3[f(x,y)]$$

$\therefore$ f is a homogeneous function of order 3.

(i.e) logz is a homogeneousfunction of order 3.

By Euler's theorem

$$x.\frac{\partial(\log z)}{\partial x} + y.\frac{\partial(\log z)}{\partial y} = 3\log z$$

$$x.\frac{1}{z}.\frac{\partial z}{\partial x} + y.\frac{1}{z}.\frac{\partial z}{\partial y} = 3\log z$$

$$x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = 3z.\log z$$

8.) If $u = \log\left(\frac{x^5 - y^5}{x^2 + y^2}\right)$ show that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 3$

Solution:

$$u = \log\left(\frac{x^5 - y^5}{x^2 + y^2}\right)$$

$$\therefore e^u = \left(\frac{x^5 - y^5}{x^2 + y^2}\right) = z \qquad \text{(say)}$$

$$\therefore z(x, y) = \frac{x^5 - y^5}{x^2 + y^2}$$

$$z(tx, ty) = \frac{(tx)^5 - (ty)^5}{(tx)^2 + (ty)^2} = \frac{t^2(x^5 - y^5)}{t^2(x^2 + y^2)}$$

$$= t^3\left(\frac{x^5 - y^5}{x^2 + y^2}\right) = t^3 z(x, y)$$

$\therefore z(x, y)$ is a homogeneous function of order 3.

$\therefore$ By Euler's Theorem,

$$x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = 3z$$

$$\text{i.e, } x.\frac{\partial}{\partial x}\left(e^u\right) + y.\frac{\partial}{\partial y}\left(e^u\right) = 3e^u$$

$$\text{i.e, } x.e^u\frac{\partial u}{\partial x} + y.e^u\frac{\partial u}{\partial y} = 3e^u$$

Dividing by e^u

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 3$$

EXERCISE

PART-A

1.) Find maximum (or) minimum Point of $y = 4x - 2x^2$

2.) Find maximum (or) minimum Point of $y = x^2 - 10x$

3.) Find maximum (or) minimum Point of $y = \text{Cos}x$

4.) When x=3 and $\frac{d^2y}{d^2x} = 4x - 3$ verify the point is maximum or minimumpoint

5.) When x=2 and $\frac{dy^2}{dx^2} = 2x + 1$ verify the point is maximum or minimum point

6.) If x=-1 is the minimum point of $y = x^2 + 2x + 3$ what is the minimum value?

7.) If $x = \frac{1}{2}$ is the minimum point of $y = x - x^2$ what is the minimum value?

8.) Find the maximum value of $y = 2x - x^2$

9.) Find the maximum value of $y = x - \sqrt{x}$

10.) Find the maximum value of $y = x - x^2$

11.) Write down the conditions for maximum value of $y = f(x)$ at $x = a$

12.) Write down the conditions for maximum value of $y = f(x)$ at x=a

13.) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following .

a.) $u = 5x^3 - 2x + 6y^2 - 18$

b.) $u = 5\sin x + 4\tan y$

c.) $u = 4e^x + 7.\sin y + 8$

d.) $u = x^2.\sin y$

e.) $u = y^2.\sec x$

f.) $u = \sin 4x.\text{Cos}2y$

g.) $u = \tan^{-1}\left(y/x\right)$

h.) $u = \log\left(e^u + e^y\right)$

i.) $u = xy + \sin xy$

j.) $u = x^2.\mathrm{Cos}\left(\frac{x}{y}\right)$

14.) Find the order of the following homogeneous equation

a.) $u = x^3 + y^3 + z^3$ (b) $u = \frac{x^{1/2} + y^{1/2}}{x^{1/4} + y^{1/4}}$

c.) $u = \frac{x^2\left(x^2 - y^2\right)^3}{\left(x^2 - y^2\right)^3}$ (d) $u = 3x^2 + 6xy + y^2$

e.) $u = \frac{1}{x^2 + xy + y^2}$

PART-B

1.) Find the maximum and minimum values of the following functions:

(a) $y = 2x^2 + 3x^2 - 36x + 1$

(b) $y = 2x^2 - 15x^2 + 36x + 18$

(c) $y = 2x^3 - 3x^2 - 12x + 5$

(d) $y = 2x^3 - 21x^2 + 72x + 1$

(e) $y = x^3 - 6x^2 + 9x + 1$

(f) $y = 4x^3 - 18x^2 + 24x - 7$

(g) $y = x^3 - 3x^2 - 9x + 2$

(h) $y = (x-1)(x+1)^2$

(i) $y = x + \frac{4}{x+2}$

(j) $y = \frac{1 - x + x^2}{1 + x + x^2}$

2.) Find the numbers whose sum is 10 and whose product is the maximum.

3.) Find two numbers whose product is four and the sum of square is a minimum.

4.) ABCD is a rectangle in which AB= 9cm and BC=6cm Find a point P in CD such that $PA^2 + PB^2$ is a minimum

5.) The perimeter of a rectangle is 100 meters. Find the side when the area is maximum.

6.) A square sheet of metal has each edge 8 cm long. Equal squares are cut off at each of the Corner, and the flaps are then folded up to form an open rectangular box. Find the side of the square cut off so that the volume of the box may be a maximum.

7.) Find the maximum value of $\frac{\log x}{x}$ for positive value of x.

8.) Find the minimum area of a rectangle inscribed is a semicircle.

9.) Show that the maximum rectangle inscribed in a circle is a square.

10.) Find the fraction, the difference between which and its square is a maximum

11.) Find 2 numbers whose product is four and the sum of their square is a minimum.

12.) Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the following functions

(a.) $u = \frac{x}{y^2} - \frac{y}{x^2}$ (b.) $u = \tan^{-1}\left(\frac{y}{x}\right)$

(c.) $u = x^3 + 3xy^2 + y^3$ (d.) $u = x.\mathrm{Siny} + y.\mathrm{Sinx}$

13.) Verify Euler's theorem for the functions.

(a.) $u = x\log\left(\frac{y}{x}\right)$ (b.) $u = \sin\left(\frac{x-y}{x+y}\right)^{1/2}$

(c.) $u = \frac{1}{x^2 + xy + y^2}$ (d.) $u = x^3 - 2x^2y$

(e.) $u=\dfrac{x^3+y^3}{x^2+y^2}$ (f.) $u=\dfrac{x^{1/3}+y^{1/3}}{x^2+xy+y^2}$

(g.) $u=\dfrac{1}{\sqrt{x^2+y^2}}$ (h.) $u=x^3.\sin\left(\dfrac{y}{x}\right)$

(i) $u=\dfrac{x^2+y^2}{y+x}$

14.) If $u=\sqrt{x^2+y^2}$, Show that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=u$

15.) If $u=\sin^{-1}\left(\dfrac{y}{x}\right)$ and $x\neq 0$ Prove that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=0$

16.) If $u=\sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, Prove that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=0$

17.) If $u=\log\left(\dfrac{x^3+y^3}{x+y}\right)$, Prove that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=2$

18.) If $u=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$, Prove that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=0$

19.) If $u=\sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, Prove that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=\tan u$

20.) If $u=\dfrac{x^2+y^2}{\sqrt{x+y}}$, Prove that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=\dfrac{3}{2}u$

21.) If $u=\sin^{-1}\left(\dfrac{\sqrt{x}+\sqrt{y}}{x+y}\right)$, Prove that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=\dfrac{-1}{2}\tan u$

22.) If $u=\tan^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$, Prove that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\sin 2u$

23.) If $u=\dfrac{x^3y^3}{x^3+y^3}$, Show that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=3u$

24.) If $u = \sin^{-1}\left(\dfrac{x^3 - y^3}{x + y}\right)$, Show that $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = 2\tan u$

25.) If $\dfrac{x^4 + y^4}{x^2 + y^2}$, Show that $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = 2u$

26.) If $u = x^2 + y^2 \text{Sin}\left(\dfrac{y}{x}\right)$, Show that $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = 4u$

27.) If $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, Show that $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = 0$

ANSWERS
PART-A

(1) 1 (2) 5 (3) 0 (4) 2

(5) $\dfrac{1}{2}$ (6) 1 (7) $\dfrac{-1}{4}$ (8) $\dfrac{1}{4}$

(13) (a) $\dfrac{\partial u}{\partial x} = 15x^2 - 2, \dfrac{\partial u}{\partial y} = +12y$

(b) $\dfrac{\partial u}{\partial x} = 5\cos x, \dfrac{\partial u}{\partial y} = 4\sec^2 y$

(c) $\dfrac{\partial u}{\partial x} = 4.e^x, \dfrac{\partial u}{\partial y} = 7\cos y$

(d) $\dfrac{\partial u}{\partial x} = \sin y.2x, \dfrac{\partial u}{\partial y} = x^2.\cos y$

(e) $\dfrac{\partial u}{\partial x} = y^2 \sec.\tan x, \dfrac{\partial u}{\partial y} = \sec x.2y$

(f) $\dfrac{\partial u}{\partial x} = \cos 2y.\cos 4x.4, \dfrac{\partial u}{\partial y} = \sin 4x - \sin 2y.2$

(g) $\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}.y.\frac{-1}{x^2}$

(h) $\frac{\partial u}{\partial x} = \frac{1}{e^x + e^y}.e^x$

(i) $\frac{\partial u}{\partial x} = y + \cos xy.y$

(j) $\frac{\partial u}{\partial x} = x^2 - \sin\left(\frac{x}{y}\right).\frac{1}{y} + \cos\left(\frac{x}{y}\right)2x$

(14) (a) 3 (b) $\frac{1}{4}$ (c) 2

(d)2 (e) -2

PART-B

1) a) 82,- 43 b)46,45

(c)12, - 15 (d) 82, 81

(e)5,1 (f)3, 1

(g)-25,7 (h) $\frac{-500}{27}, \frac{7}{3}$

(i) -6, 2 (j)3, $\frac{1}{3}$

2) 5,5 3) 2,2 4) 4,5 5) 25

6) $\frac{4}{3}$ 7) $\frac{1}{e}$ 8) a^2 9) Nil

10) $\frac{1}{2}$ 11) 2,2

MATHEMATICS – II

MODEL QUESTION PAPER – 1

Time : 3 Hrs　　　　　　　　　　　　　　　　Max Marks : 75

PART – A

(Marks: 15 x 1 = 15)

1. ANSWER ANY 15 QUESTIONS:

1) Find the centre and radius of the circle $x^2 + y^2 + 4x - 2y + 3 = 0$

2) Find the equation of the circle with centre (-2, -4) and radius 5 Units.

3) Write down the equation of the circle with end points of a diameter (x_1, y_1) and (x_2,y_2)

4) Show that the point (5, -12) lies outside the circle $x^2+y^2-2x+2y-60=0$

5) State the condition for two circles to cut orthogonally

6) Evaluate $\underset{x \to 0}{Lt} \dfrac{\sin 2x}{3x}$

7) Find $\dfrac{d}{dx}\left\{\dfrac{1}{x^3} + 7\cos x\right\}$

8) Find $\dfrac{d}{dx}\left\{x^4 \tan x\right\}$

9) Find $\dfrac{d}{dx}\left[\cos(\log x)\right]$

10) Find $\dfrac{d}{dx}\left[\sin^{-1}\left(\sqrt{x}\right)\right]$

11) Find $\dfrac{d^2y}{dx^2}$ if y = tan x

12) Find the differential equation by eliminating constant r, from $x^2 + y^2 = r^2$

13) If A = x^2 and $\frac{d}{dx}=2$ find $\frac{dA}{dt}$ when $x=5$

14) If the distance s given by s = $3t^2+5t+7$, find the velocity when t=3 seconds.

15) Find the slope the tangent to the curve $y=x^2-5x+2$ at the point (1 ,-2).

16) Find the slope of normal to the curve $y=\sqrt{x}$ at (4,-2).

17) Show that the function $y=4x-x^2+7$ is the maximum at x=2.

18) If $u = x^3 + 5x^2y+y^3$ find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

19) If $u = \log(x^2+y^2)$ find $\frac{\partial u}{\partial x}$

20) State Euler's Theorem.

PART – B

(Answer Any TWO subdivisions in each question)

All Questions carry Equal Marks

5x12=60

21 a) Find the equation of the circle passing through the point (-9,1) and having centre at (2,5)

b) Find the equation of the circle passing through the points (0,1),(2,3)and (-2,5)

c) Find the equation of the tangent at (5, -2) to the circle $x^2 + y^2 - 10x - 14y - 7 = 0$

22.a) Show that the circles $x^2+y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other.

b) Evaluate $\underset{x\to 3}{Lt} \frac{x^5-243}{x^3-27}$

c) Differentiate the following:-

(i) $y = e^x \log x \sin x$. (ii) $y=\frac{x^2+\sin x}{x-\cos x}$

23.a) Find $\frac{dy}{dx}$ if (i) $y = \log(\sec x+\tan x)$ (ii) $ax^2+2hxy+by^2 = 0$

b) Find $\frac{dy}{dx}$ if (i) $y = \cos^{-1}\frac{1-x^2}{1-x^2}$ (ii) $x=a(t+\cos t)$, $y=a(1+\sin t)$

c) if $y = x^2 \cos x$, prove that $x^2 y_2 - 4xy_1+(x^2+6)y=0$

24.a) The radius of a sphere is increasing at the rate of 1cm/sec. How fast the volume will be increasing when the radius is 4cm

b) A missile is fired from the ground level rises x meters vertically upwards in time 3 seconds and $x = 100t - \frac{25}{2}t^2$ .Find the initial velocity and maximum height of the missile

c) Find the equation of the tangent and normal to the curve $y=x^2-x+1$ at (2,3).

25.a) Find the maximum and minimum values of

$2x^3 - 15x^2 + 36x + 18$

b) If $u = x^3 - 2x^2y + 3xy^2 + y^3$, Find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$

c) If $u = \frac{x^3 - y^3}{x+y}$ Prove that $x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2u$

MATHEMATICS - II

MODEL QUESTION PAPER - 2

Time: 3 Hours max. Marks: 75

PART - A

I. Answer any 15 Questions:

1. Find the equation of the circle with centre (2,0) and radius 10 units
2. Find the centre and radius of the circle $x^2 + y^2 = 4$
3. Find the equation of the circle with the points (1, -1) and (2, 2) joining as diameter.
4. Find the length of tangent from the point (5,7) to the circle $x^2+y^2-6x+10y-11=0$.
5. Show that this circles $x^2+y^2-10x+4y-13=0$ and $x^2+y^2-10x+4y-19=0$ are concentric circles.
6. Evaluate $\underset{x \to 3}{\mathrm{Lt}} \dfrac{x^3 - 3^3}{x - 3}$
7. Find $\dfrac{d}{dx}$ if $y = \dfrac{1}{x^2} + \dfrac{2}{x} + \dfrac{3}{2}$
8. Find $\dfrac{d}{dx}$ If $y = e^x \log x$
9. Find $\dfrac{d}{dx}$ If $y = \cos^4 x$
10. Find $\dfrac{dy}{dx}$ if $y = \tan^{-1}(x^2)$
11. Find $\dfrac{d^2y}{dx^2}$ if y = sin (2x)

12.. Find the differential equation by eliminating the constants from $y = ax^2+b$

13. If $V=a^3$ and $\dfrac{da}{dt}=1$, find $\dfrac{dy}{dt}$ when a = 5

14. If $S=ae^{t}+be^{-t}$, show that acceleration is always equal is to distance

15 If the distance time formula is given bye $s=2t^{3}-5t^{2}+7t-4$, find the initial velocity.

16. Find the slope of the normal to the curve $y = x^{2} + 7x$ at (1,8)

17. Find the minimum value of $y = x^{2} + 4x + 1$

18. If $u = x^{3}+x^{2}y+2xy^{2}-y^{3}$ find $\frac{\partial u}{\partial y}$

19. If u = tan (ax+by) find $\frac{\partial u}{\partial y}$

20. Show that $\frac{x^{2}+y^{2}}{x-y}$ is homogeneous. State the order of the function.

PART - B

Answer any TWO sub division from each question:

All Questions carry Equal Marks

5 X 12 = 60

21.a) Find the equation of the circle, two of whose diameters are $x + y = 6$ and $x + 2y = 4$ and whose radius is 10 Units.

b) Find the equation of the circle passing through (0, 1) and (4, 3) and having its centre on the line $4x - 5y - 5 = 0$

c) Find the equation of the tangent at (4, 1) on the circle $x^{2} + y^{2} - 2x + 6y - 15 = 0$

22 a) Find the equation of the circle which passes through the origin and cuts Orthogonally with circles $x^{2} + y^{2} - 8y + 12 = 0$ and $x2 + y2 - 4x - 6y - 3 = 0$

b) Evaluate $\underset{x \to 0}{\text{Lt}} \quad \frac{\sin 10x}{\sin 7x}$

c) Find $\frac{dy}{dx}$ if (i) $y=\frac{a}{x^{2}}+\frac{b}{x^{3}}+\frac{c}{x}$

(ii) $Y = (X^{2}-5)$ Cos x log x

23 a) Find $\frac{dy}{dx}$ if (i) $y = \sin(e^x \log x)$

(ii) $x^3+y^3 = 2axy$

b) Find $\frac{dy}{dx}$ if (i) $y = \tan^{-1}\frac{2x}{1-x^2}$

(ii) $x = at^2$, $y = 2at$

c) if $y = a\cos(\log x) + b\sin(\log x)$ prove that $x^2y_2+xy_1+y=0$

24 a) The base radius and height of a conical funnel are 4cm and 20cm respectively. Water is running out of the funnel at 2cc/sec. Find the rate at which the level of water is decreasing when the level is 10cm.

b) If the distance time formula is given by $s=2t^3-15t^2+36t+7$, find the time when the velocity becomes zero.

c) Find the equation of the tangent and normal to the curve $y = 6 + x - x^2$ at (2, 4)

25 a) Find the maximum and minimum value of $y = 4x^3-18x^2+24x-7$.

b) If $u = \log(x^2+y^2)$ find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

c) If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$